Workshop: Optimal Design
Laboratoire de Mécanique des Solides
Ecole Polytechnique Palaiseau France
November 26-28, 2003

Concurrent Design and Material Selection for Trusses

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Abstract
The choice of the best material (meaning the least mass of material carries the given loads safely) is different for axially loaded tension and compression members. Using two materials in trusses, one for tension members and the other for compression members, is not a good solution because of manufacturing restrictions, potential for galvanic corrosion, and other incompatibility problems. In this paper, we develop a criterion for choosing a single material for a truss with given loads in all its members. These loads change when the geometry of the truss changes, which in turn changes the choice of the best single material. Then a question arises: what is the best choice of a single material, truss geometry, and shapes and sizes of cross-sections of the members that maximizes the stiffness, minimizes the material weight or cost, and carries the loads safely? Here, we present criteria and a method for concurrent design and material selection using simple triangulated trusses as examples.

1. Introduction

To optimize the performance and cost, the task of a designer is to determine the geometry of component(s) and material(s) for them under some functional requirements and constraints. The term "geometry" is used here to imply the connectivity and layout (i.e., topology) of components as well as the shapes and sizes of the components. When the designer commits to certain materials, there are techniques to optimally determine their geometry. Likewise, when the geometry is chosen, there are techniques to select the best material(s). Fixing one in general influences the optimality of the other. Concurrently determining the optimal geometry and selecting the best material(s) remains an open issue in many design problems. This can be best explained in the case of load bearing structures.

Structural designers usually want to minimize the weight (or cost) and maximize the stiffness for given loads and available space. When a material is chosen, there are techniques to optimally determine the geometry including topology, shape, and size (see, for example, a book by M. P. Bendsoe and O. Sigmund (2003)). On the other hand, for components of chosen geometry, most suitable materials can be systematically identified (see, for example, a book by M. F. Ashby (1999)). The criteria for material selection for a structural component depend upon the type of loading on it such as tension, compression, bending, twisting, etc. But a structure as a whole may consist of many components with different types of loads in them. Allocating a different material that is best for each component is not a good solution because it makes manufacturing difficult and may lead to galvanic corrosion and other incompatibility problems. Thus, there is a need to develop criteria for choosing the best single material for the entire structure. This is the first issue addressed in this paper.

The criteria developed for selecting a *best* single material for all components depends on the type(s) of loading in the individual components. But the type of loading in components will in general change as the geometry (topology or layout, in particular) of the structure changes implying that the choice of the best material changes with geometry. Therefore, fixing the geometry and then selecting the best material or determining the geometry for an arbitrarily selected material are not likely to be truly optimal. A method for achieving the optimal material-geometry combination is the second issue addressed in this paper.

In the remainder of this paper, the two issues outlined above are further elaborated using simple planar trusses as examples, and weight and cost of material, stiffness, and strength as design objectives. So, the next section considers a three-bar truss to present the criterion for selecting a single material for the entire truss. Following this, the method for concurrent design and material selection for this truss is described. The two ideas are generalized for other trusses in the subsequent section with concluding remarks in the final section.

2. Material selection for a three-bar truss

The members that make up truss structures are not all loaded in the same way: some carry tension, and others compression. Consider the simplest form of a statically determinate, simply supported truss consisting of three bars with vertical and horizontal loads applied at the unsupported vertex. The truss and its parameters are shown in Fig. 1 and Table 1. The angle θ determines the geometric configuration of the truss, L its size, and F the loading acting on it.

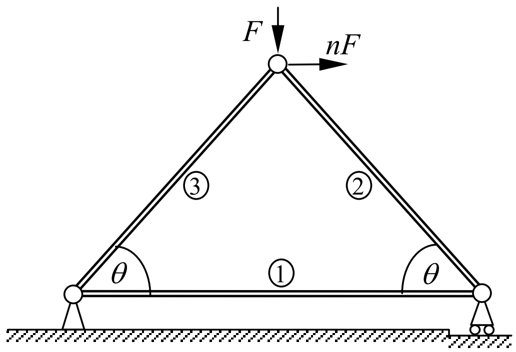

Fig. 1 Schematic of the truss

Table 1 Truss parameters

Truss member	*Length*	*C/S area*	*Internal force*
1	$2L\cos(\theta)$	A_1	P_1
2	L	A_2	P_2
3	L	A_3	P_3

For static equilibrium of the entire truss, the horizontal reaction force at the left support is simply $(-nF)$. The vertical reactions at the left and right supports, R_l and R_r respectively, are given by:

$$R_l = \frac{F(1-n\tan\theta)}{2};\quad R_r = \frac{F(1+n\tan\theta)}{2} \tag{1}$$

Further, the static equilibrium at each vertex gives the internal forces as follows such that a positive numerical value indicates tension, and negative value compression.

$$P_1 = \frac{F(1+n\tan\theta)}{2\tan\theta};\quad P_2 = -\frac{F(1+n\tan\theta)}{2\sin\theta};\quad P_3 = -\frac{F(1-n\tan\theta)}{2\sin\theta} \tag{2}$$

Since slender tension members fail by yielding, and compression members by buckling, we determine the assumed *square* areas of cross-sections as follows where the subscripts t and c indicate tension and compression respectively.

$$A_t = \frac{P_t}{\sigma_y};\quad A_c = \sqrt{\frac{12L_c^2P_c}{\pi^2E}} \tag{3}$$

The overall mass of the structure can then be given by

$$m = \left\{\sum_{i=1}^{N_t}(P_{t_i}L_{t_i})\right\}\frac{\rho}{\sigma_y} + \left\{\sum_{j=1}^{N_c}\frac{L_{j_c}^2}{\pi}\sqrt{12\left|P_{j_c}\right|}\right\}\frac{\rho}{E^{1/2}} = \psi_t\frac{\rho}{\sigma_y} + \psi_c\frac{\rho}{E^{1/2}} \tag{4}$$

where N_t and N_c indicate the number of tension and compression members respectively. The above equation includes two material indices, viz., (ρ/σ_y) and $(\rho/E^{1/2})$. The values of coefficients, ψ_t and ψ_c, which multiply the respective material indices, are determined by the type of loading (tension or compression) and its magnitude that exists in each member of the truss. If these values are known, the best material for minimum mass can be chosen from a material database. It should be noticed that it is the ratio of the two coefficients that decides the choice of the best material rather than their individual numerical values. A design index γ is now defined as follows to facilitate drawing of the contours of constant mass in the plot of (ρ/σ_y) against $(\rho/E^{1/2})$ and thereby selecting the most suitable material.

$$\text{Design index} = \gamma = \frac{\psi_c}{\psi_t} \quad (5)$$

This design index indicates the relative weights of the two material indices in Eq. (4). In order to see how this is useful, consider a performance measure $f = C_1 M_1 + C_2 M_2$, which we want to minimize. This represents a straight line in a plot of M_1 vs. M_2 for given values of f and (C_2 / C_1). Fig. 2a shows a number of such *iso-performance* lines drawn for the fixed ratio of $C_2 / C_1 (= 4.27)$, and different values of f. As we move down, the value of f decreases. Imagine that all materials from a database are drawn as ovals in this plot according to their range of respective material indices, i.e., maximum and minimum values. Now, by moving the f-line downwards, we can identify which material leads to its minimum value. If the same is plotted on a log-log plot, which is often necessary as properties of different materials differ by a few orders of magnitude, the f-line becomes a curve as shown in Fig. 2b (see Section 9.6 in Ashby (1999) for further description). The CES® (2003) software allows this visualization on linear and log plots for any combination of material indices. Now, we return to our example and show how the design index is useful.

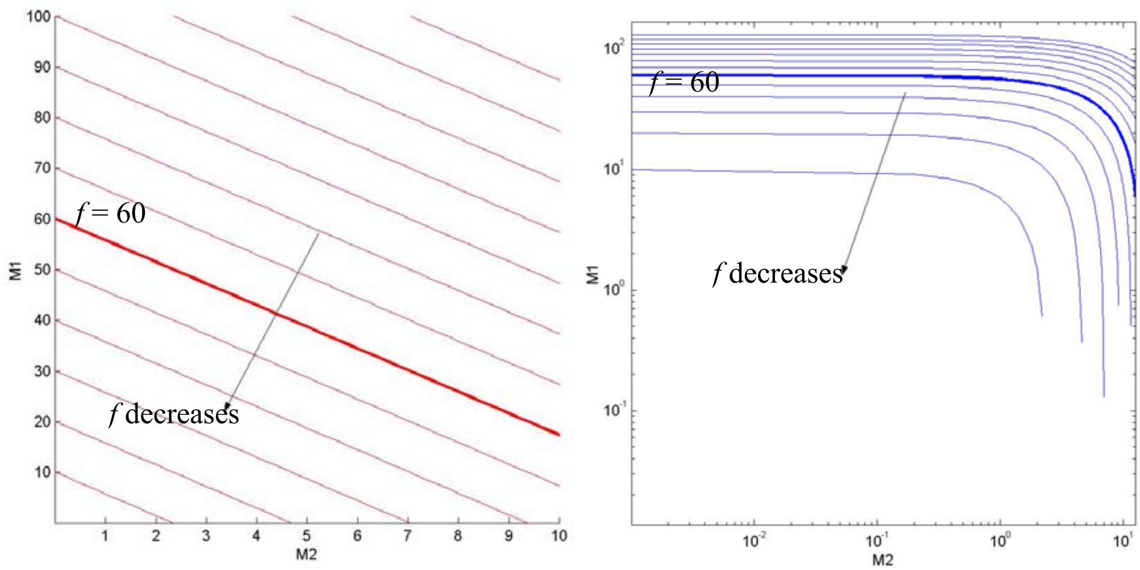

Fig. 2 Visualizing the minimization of a performance measure f that is a linear combination of two material indices, M_1 and M_2 for a given ratio of their weights. (a) linear plot (b) log-log plot

Considering only metals in the database, in Fig. 3, the iso-performance curves with the least possible mass in each case are shown for three different values of γ = 0.01, 1, 100. Up to γ = 1, aluminum-boron composite turns out to be most probable choice. For higher values, beryllium alloys dominate. Titanium alloys seem to be the next best candidates, among a few others. All of these are expensive materials to build trusses. Thus, we need to include the material cost. For this, we can use the same index but plot it in the graph of $\left(\text{Price} * \rho / \sigma_y\right)$ against $\left(\text{Price} * \rho / E^{1/2}\right)$. Then, as shown in Fig. 4, low alloy steels emerge as the best choice until γ = 1. For larger values of γ, cast irons dominate. This is sensible because large values of γ indicates that more members in the truss are under compression. In both the figures, it helps to examine other desirable materials by translating the iso-performance curves.

Next, all polymer materials in the database were considered. Figs. 5 and 6 show possibilities when cost is not considered and when it is considered, respectively. When cost is not a criterion (Fig. 5), expensive fiber-reinforced polymers appear. With cost included, more reasonable polystyrene, polypropylene, etc., appear as the most suitable choices. The same was also done considering all materials in the database (figures not shown). When cost is included, concrete appears as one of the best materials.

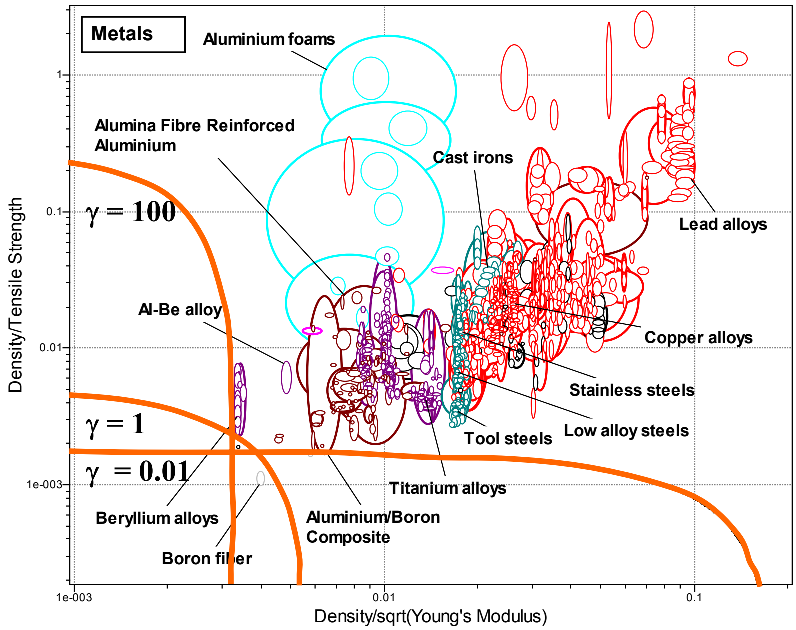

Fig. 3 Material selection for trusses considering only metals. Since price is not included, expensive alloys and fiber composites turn out to be the most suitable materials to minimize the mass.

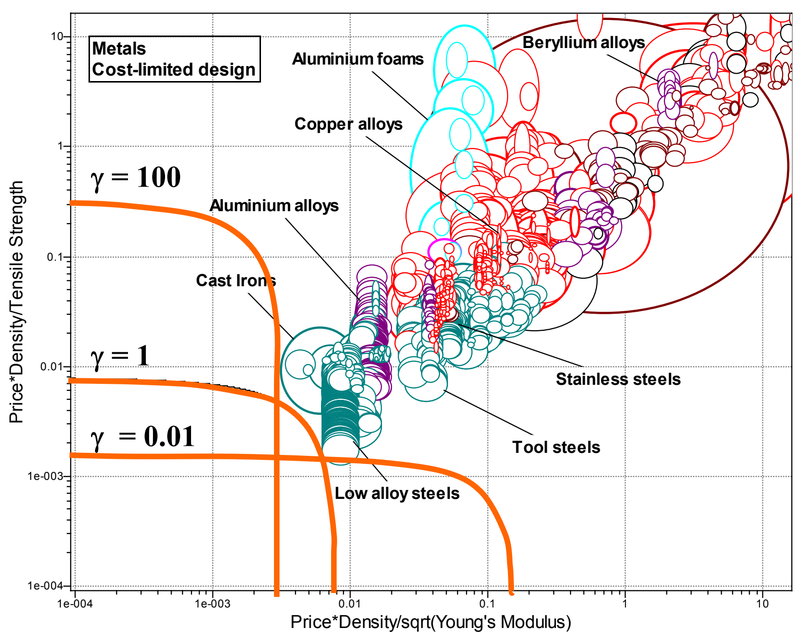

Fig. 4 Material selection for trusses considering only metals but cost included. Low alloy steels and cast irons now emerge as the best candidates.

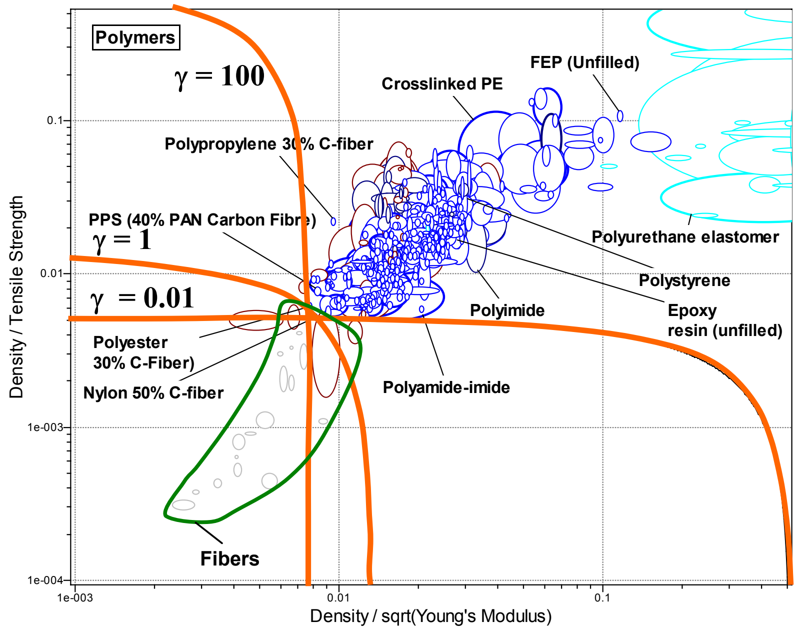

Fig. 5 Material selection for trusses considering only polymers. Fiber reinforced polymers appear to be the best because cost is not a criterion. Fibers themselves, enclosed in the green envelope, are ignored.

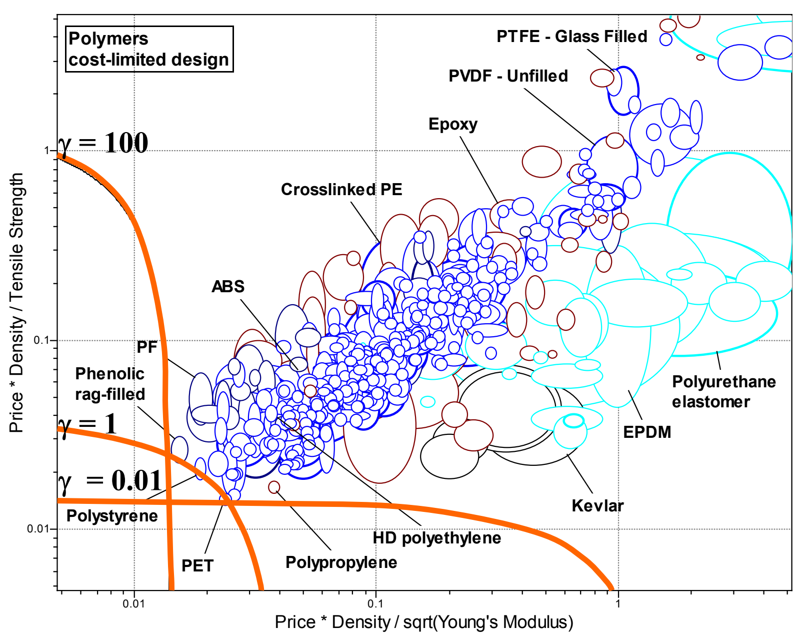

Fig. 6 Material selection for trusses considering only polymers. With cost included, polypropylene, polystyrene, etc., emerge as the most suitable materials.

It is thus clear that a single material can be selected even though the individual members in the truss may indicate different best material choices. The Index, γ, developed in this section can also be applied to other similar problems.

Consider a simple extension to member sections that are not square but are shaped to increase—or if desired, to reduce—bending stiffness of compression members. We define a *shape factor* S as

$$S = \frac{12I}{A^2} \tag{6}$$

where I is the smaller second moment of inertia of the section and A is the area of its cross-section. Note that $S = 1$ for a square cross-section; efficiently shaped sections (tubes, I-sections, etc.) have values as high as 30. The areas of tensile and compressive members, previously given by Eq. (3) now become:

$$A_t = \frac{P_t}{\sigma_y}; \quad A_c = \sqrt{\frac{12L_c^2 P_c}{\pi^2 SE}} \tag{7}$$

and the values of ψ_t and ψ_c become

$$\psi_t = \left\{ \sum_{i=1}^{N_t} (P_{t_i} L_{t_i}) \right\}; \quad \psi_c = \left\{ \sum_{j=1}^{N_c} \frac{L_{j_c}^2}{\pi} \sqrt{\frac{12\left|P_{j_c}\right|}{S}} \right\} \tag{8}$$

Thus, the procedure described above for choosing the single best material now can also select the best material-shape combination for all members in the truss.

Even in this simple example, it is interesting to note that γ depends on the scale factors of size (L) and load (F), and geometry (θ). In particular, γ varies linearly with L, and bears an inverse relationship with $F^{1/2}$. Thus, for a given configuration and loading, the choice of the material is not affected if the ratio $\left(F / L^2\right)$ is held constant—a fact well known in structural optimization (F. R. Shanley, 1953). It should also be noted that the dependence of γ on θ and n is not straightforward. This is explored further to suggest a scheme for integrating material selection into the process of determining the geometry of the truss, i.e., θ in this case.

2. Concurrent design and material selection for the three-bar truss

Using the expressions presented above, it is easy to plot the variation of γ as a function of θ while keeping the rest of the parameters fixed. This means that we are trying to determine how the geometric variable (θ) affects γ, which in turn affects the material (and shape) selection. Figs. 7a-7c show that this dependence is complex and is certainly not monotonic. A sharp change in these graphs is a result of one or more members of the truss making a transition from tension to compression or vice versa. This implies that changing the geometry of the truss could strongly affect the design index used to choose the most suitable material. The question then arises as to how one would determine the best geometry in conjunction with the most suitable material (and cross-section shape). This requires an objective criterion. In the case of the truss, stiffness of the truss is a reasonable criterion to work with. The problem of *concurrent design and material selection* now becomes equivalent to choosing the value of θ such that the truss becomes the stiffest among all the possible geometric cases where the most suitable material is chosen in each case. Thus, it becomes an optimization problem for the *stiffest and strong truss made of the cheapest material and most suitable cross-section shape for a given loading and size*.

In order to solve this problem using gradient-based optimization methods, a smooth function of relevant material properties as a function of geometric design variables (θ in the present example) is necessary. This could be done with the help of the CES® software although it requires tedious effort (which could be potentially automated). An

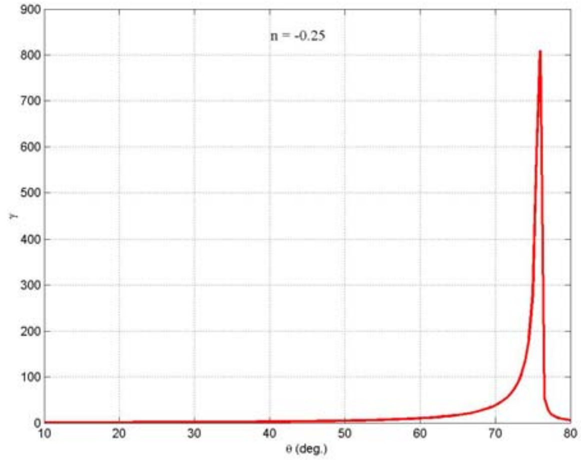

Fig. 7a γ **vs.** θ **for** n **= -0.25**

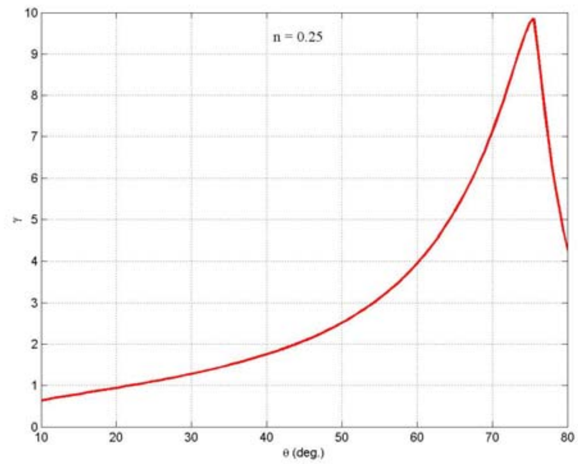

Fig. 7a γ **vs.** θ **for** n **= 0.25**

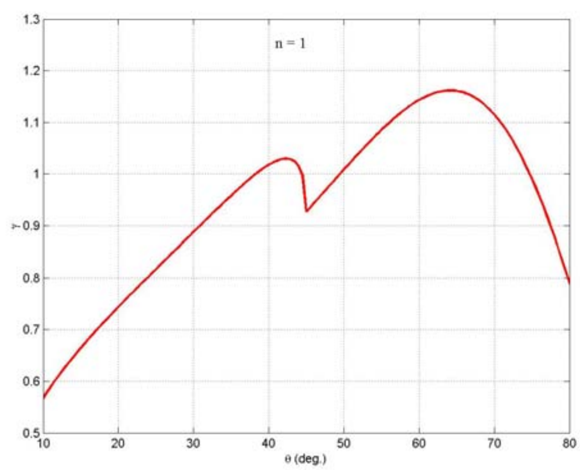

Fig. 7a γ **vs.** θ **for** n **= 1**

example of this is shown in Fig. 8 where the density (ρ) and the Young's modulus (E) are shown as functions of γ. The data points in these plots were hand-picked by examining the $\left(\text{Price} * \rho / \sigma_y\right)$ against $\left(\text{Price} * \rho / E^{1/2}\right)$ plots for each value γ so as to minimize the mass. If all materials are considered, the best material for different ranges of γ are shown in Fig. 8. Since cost was included, reasonable materials like low alloy steel, concrete, and wood turn out to be the desirable choices. If only polymers are considered, the best materials are shown in Fig. 9. Fitting a suitable smooth curve to this point data facilitates integration of material selection into stiffness optimization of the truss. A look-up table where best materials are arranged *a priori* for each value of γ, can also be used. This latter approach is used next for simplicity leaving the other for future work.

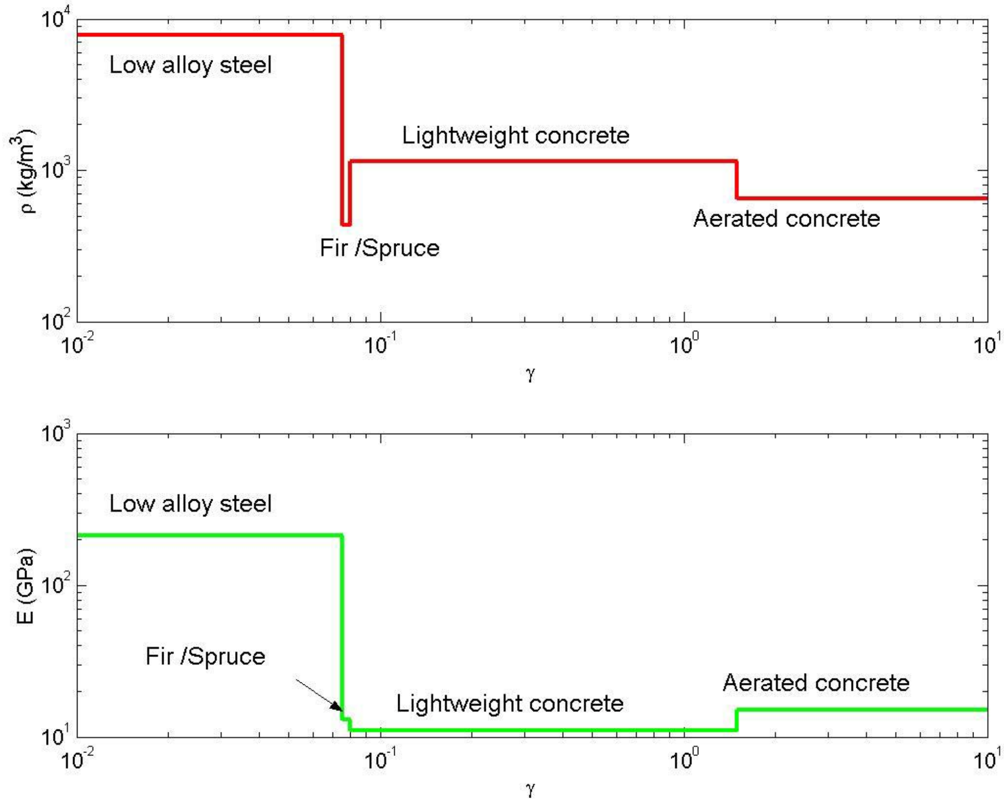

Fig. 8 Properties relevant to stiffness optimization from the best materials selected for each value of γ. Mean values are shown instead of the min/max values for each property.

The *mean compliance,* MC defined as the work done by the external forces acting on the structure, is often used as the objective criterion in the optimization for stiffest structures. The smaller the value MC, the stiffer the structure. Hence, we choose the geometry of truss by minimizing MC, which for the given loading and boundary conditions of the truss is given by:

$$MC = F\delta_V + nF\delta_H \tag{9}$$

where δ_v and δ_H are vertical and horizontal displacements of the vertex where the forces are applied (see Fig. 1). These displacements, derived using the Castiglianos' theorem, are as follows.

$$
\begin{aligned}
\delta_V &= \frac{L}{4E}\left\{\frac{2F\cos\theta}{A_1\tan^2\theta}(1+n\tan\theta)+\frac{F}{A_2\sin^2\theta}(1+n\tan\theta)+\frac{F}{A_3\sin^2\theta}(1-n\tan\theta)\right\} \\
\delta_H &= \frac{L}{2E}\left\{\frac{F\cos\theta}{A_1\tan\theta}(1+n\tan\theta)+\frac{F}{A_2\sin 2\theta}(1+n\tan\theta)-\frac{F}{A_3\sin 2\theta}(1-n\tan\theta)\right\}
\end{aligned}
\tag{10}
$$

The method for determining the geometric variable θ that gives the stiffest structure proceeds as follows. Given a value of θ, the design index γ is computed using Eqs. (4) and (5). This is then used to select the best material from the look-up table (or smoothened function if that approach is used). The values of this best material are used to compute the cross-section areas of each of the truss member as per the strength criterion using Eq. (3). Then, MC is computed using Eqs. (9) and (10). This is repeated for the entire range of values that θ is permitted to take. The plot of MC vs. θ indicates the minimum, that is, the stiffest structure. In an actual optimization procedure, this type of exhaustive "search" is not done; a gradient-based search algorithm is used instead. Here, we use this technique to illustrate some interesting features of this example as well as the non-smoothness of this optimization problem.

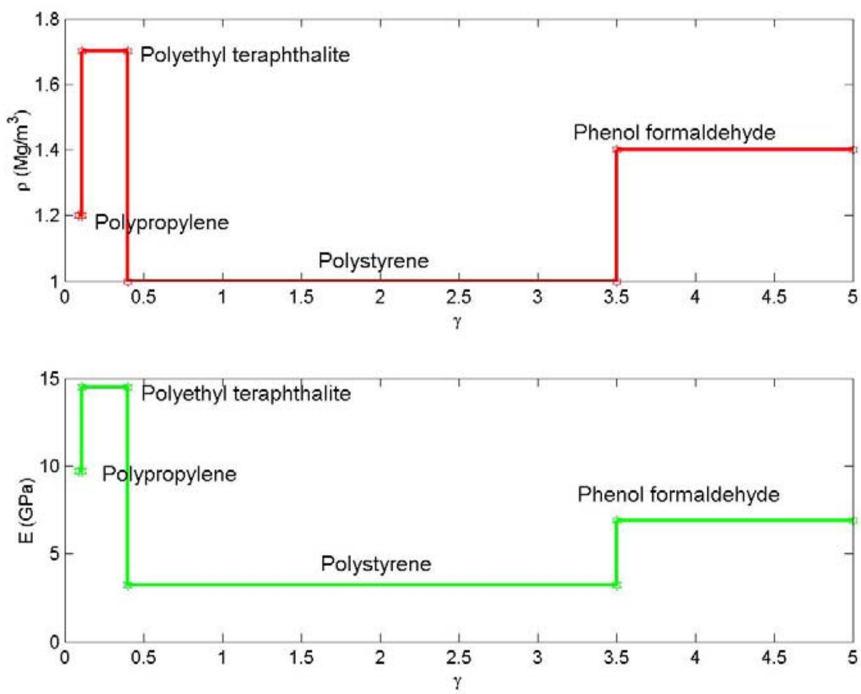

Fig. 9 Properties relevant to stiffness optimization from the best *polymer* materials selected for each value of γ. Mean values are shown instead of the min/max values for each property.

3. Results

Example 1: The results for $L=10\,m, F=1000\,N, n=-0.5$ are shown in Fig. 10. It can be seen that at the optimum value of $\theta=63^o$ corresponding to the *stiffest* truss, the total cost of material (~\$1.1) also happens to be almost close to its minimum value. As we will see in the subsequent examples, this is not always the case because we did not include a constraint on the cost although we ensured that the cheapest material was chosen for each value of θ.

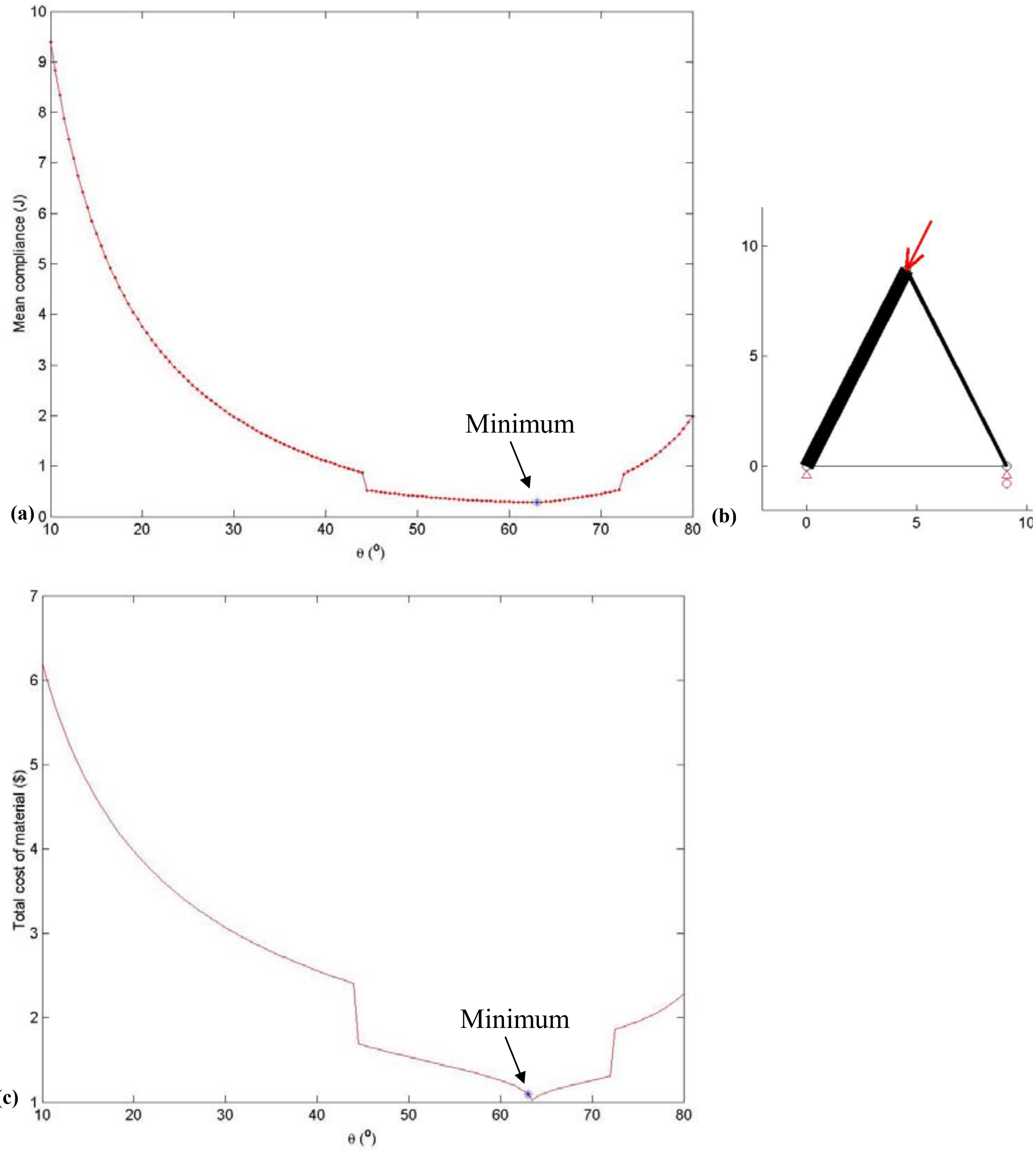

Fig. 10 Example 1 with $L = 10\,m, F = 1000\,N, n = -0.5$ (a) Stiffness measure, MC (b) Optimum truss where line-widths indicate relative proportions of cross-section on a scale of 1-10 with 10 showing the maximum width (c) Cost of material. (a) and (c) are plotted as a function of the geometric variable, θ.

<u>Example 2:</u> The optimum truss for $L = 10\,m, F = 100\,N, n = -1$, turns out to be a single-member such that the applied force is oriented along its axis to cause compression. Thus, the technique is capable of optimizing even the *layout* of the truss—*indicating the promise for topology optimization!* Here too, by coincidence, the overall cost is minimized. Furthermore, it can be noticed that the best material (aerated concrete) that has the lowest value of $\rho / \sqrt{E}$ has been chosen (as it should) given that there is only a compression member in this truss.

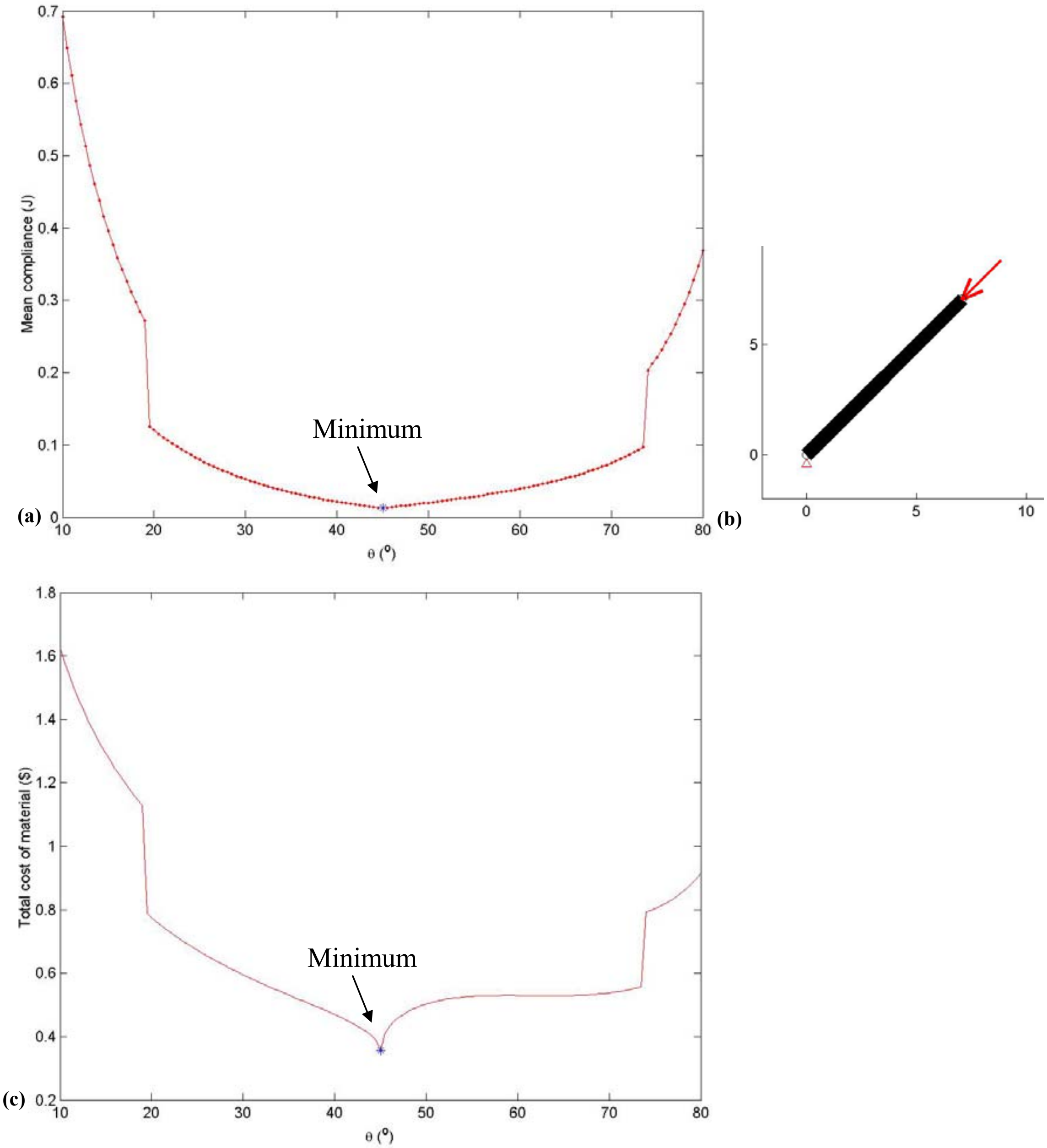

Fig. 11 Example 1 with $L = 10\,m, F = 100\,N, n = -1$ (a) Stiffness measure, MC (b) Optimum truss where line-widths indicate relative proportions of cross-section on a scale of 1-10 with 10 showing the maximum width (c) Cost of material. (a) and (c) are plotted as a function of the geometric variable, θ.

<u>Example 3:</u> For $L = 10\,m, F = -100\ N, n = 0.5$, a complete truss with all its three members present is the stiffest truss. As can be seen in Fig. 12, in this case the overall cost of the material is not a minimum. If the cost were to be minimized, then the optimum θ would be 63º for which the mean compliance is slightly larger than it is for the stiffest truss. Constrained optimization can automatically resolve these situations.

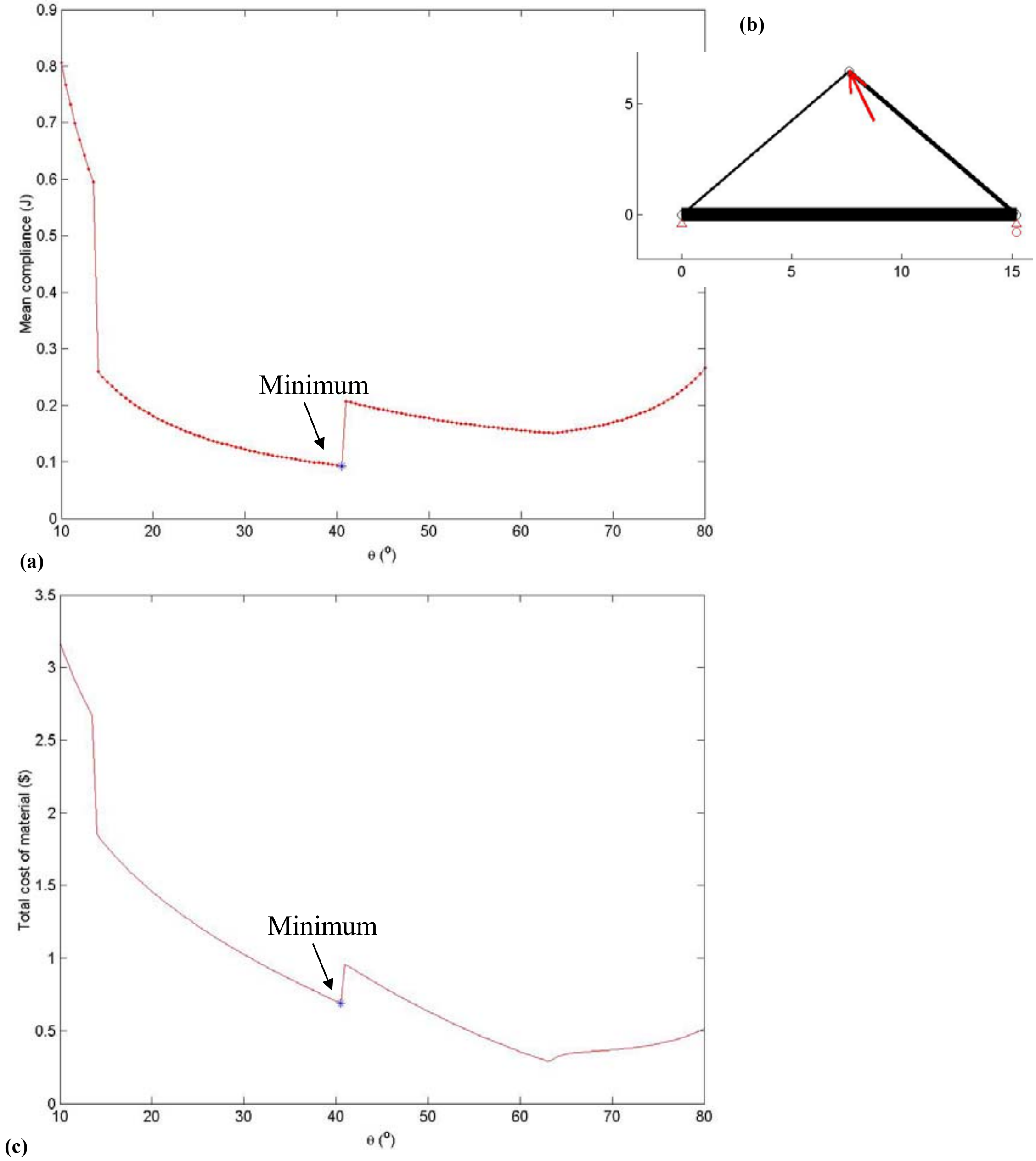

Fig. 12 Example 3 with $L = 10\,m, F = -100\ N, n = 0.5$ (a) Stiffness measure, MC (b) Optimum truss where line-widths indicate relative proportions of cross-section on a scale of 1-10 with 10 showing the maximum width (c) Cost of material. (a) and (c) are plotted as a function of the geometric variable, θ.

Example 4: This example shows that when the size of the structure is small and the force is large, the best material for the stiffest structure is low alloy steel and not aerated concrete. For $L = 0.1m, F = 1000\,N, n = -5$, the results are shown in Fig. 13. Although it may appear that the cost is compromised here (not in an absolute sense as the cost is extremely low here), it should be noted that cheapest material is chosen to provide the stiffest and strong truss. If we impose a constraint on the overall cost, then a less stiff structure will be chosen.

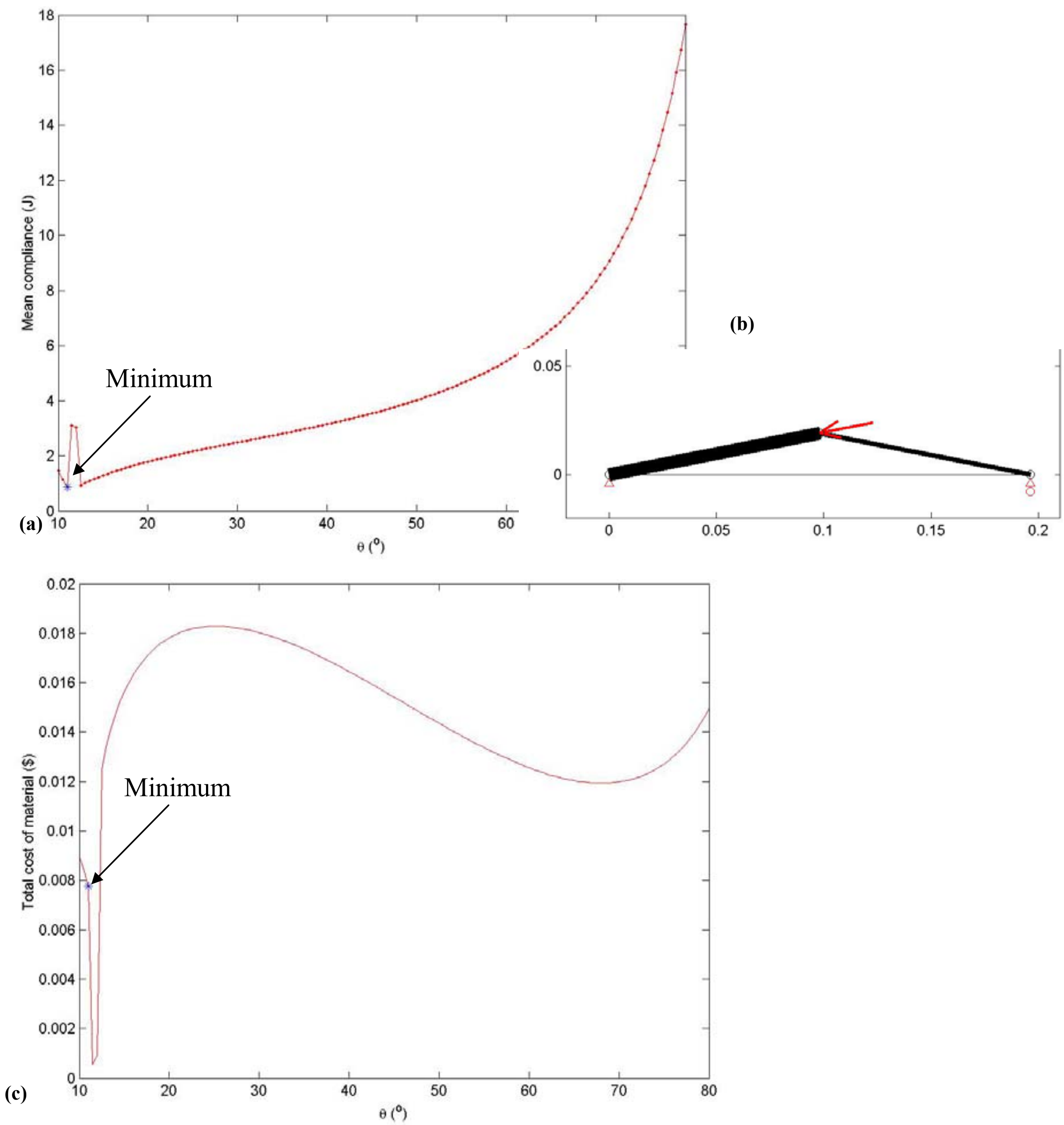

Fig. 13 Example 4 with $L = 0.1m, F = 1000N, n = -5$ (a) Stiffness measure, MC (b) Optimum truss where line-widths indicate relative proportions of cross-section on a scale of 1-10 with 10 showing the maximum width (c) Cost of material. (a) and (c) are plotted as a function of the geometric variable, θ.

4. Discussion

In spite of the availability of a high-stiffness material (low alloy steel) in the look-up table, aerated concrete was chosen in most cases because geometry dominated those designs in deciding the stiffest trusses. This is not always true as was seen in Example 4. Thus, a certain combination of loading, size, and geometry may lead to what might appear to be an unlikely choice of material. The main point of the approach presented here is that concurrent design and material selection would lead to a design that is better than either the case of a best material chosen for a given geometry or the case of a best geometry determined for a given material. Therefore, for a given loading and size, this approach enables us to choose the optimum design and the best material for it—*not before or after the design is determined but as part of the design process.*

A notable feature of the optimization problem that selects the best material concurrent with the design is that it is a *non-smooth* optimization problem. The origin of the non-smoothness is sudden transitions between different materials as the design index changes with geometry when the force in one or more members changes from compression to tension or vice versa. More importantly, the minimum lies at the non-smooth point in most cases. This is true in three out of four examples described earlier. This is problematic when smooth optimization algorithms are used. This leaves us with two options. First, we can use non-smooth optimization techniques. Second, we can smoothen the material transition by fitting *artificial material data* curves using sigmoid functions between every pair of adjacent materials in the material property vs. the design index plots.

Extension of this approach to statically indeterminate and large trusses is straightforward. It should be noted that even in statically indeterminate trusses, the internal forces in each member are independent of material properties and cross-section areas. Therefore, it is possible to compute the design index numerically. The non-smoothness should be dealt with care. Two options were noted above. Sensitivity analysis needed to compute the gradients of the objective function and constraints with respect to the design variables should now account for the changing "best" material.

5. Closure

For each component and specified loading, the best material can be identified. But the loading in components is a function of the overall geometry—layout (topology) and shape—of the ensemble. Therefore, fixing the geometry and then choosing the material, or fixing the material and determining the geometry are not truly optimal. In this paper, taking trusses as the examples, the criterion for choosing the best geometry-material combination is presented. In particular, the procedure for obtaining the stiffest truss whose members do not fail by yielding (tension members) or buckling (compression members) and have the best material and cross-section shape. The material is the same for all members in view of manufacturing and compatibility issues. Towards that, a new index that decides the single best material for all members is developed. The optimization problem involved here is inherently non-smooth, and this will be discussed in forthcoming publications.

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