

Topological Synthesis of Compliant Mechanisms Using Multi-Criteria Optimization

Compliant mechanisms are mechanical devices that achieve motion via elastic deformation. A new method for topological synthesis of single-piece compliant mechanisms is presented, using a "design for required deflection" approach. A simple beam example is used to illustrate this concept and to provide the motivation for a new multi-criteria approach for compliant mechanism design. This new approach handles motion and loading requirements simultaneously for a given set of input force and output deflection specifications. Both a truss ground structure and a two-dimensional continuum are used in the implementation which is illustrated with design examples.

Introduction

Compliant mechanisms are a relatively new breed of mechanical devices in which elastic deformation is intended as a source for motion. In designing conventional rigid-link mechanisms, elastic deformation is usually considered undesirable. However, compliant mechanisms are designed to be intentionally flexible. This flexibility allows motion, and hence the ability to perform useful work. Figure 1 shows a compliant crimping mechanism made out of polyethylene using a CNC machine (Ananthasuresh, 1994a). This single-piece compliant mechanism undergoes elastic deformation when subjected to the applied force, \( F_a \), in order to achieve the desired output motion, \( \Delta \). Single-piece, joint-less, distributed compliant mechanisms such as this offer the considerable advantage of single-piece construction and simple manufacture over conventional rigid-link mechanisms (Ananthasuresh and Kota, 1995).

Systematic methods have been developed to synthesize and design compliant mechanisms. Ananthasuresh et al. originally developed a continuum-based approach which uses the techniques of structural optimization and the homogenization method (Ananthasuresh, 1994a; Ananthasuresh et al., 1993, 1994b, 1994c). Other efforts aimed at using structural optimization techniques to design mechanisms have been developed by Sigmund (1995) and Larsen et al. (1996). The methods developed by Midha and his associates use kinematic techniques such as graph theory (Murphy et al., 1993) and Burmester theory (Mettlach and Midha, 1996), as well as a pseudo rigid-body model (Howell and Midha, 1994), where the compliant mechanism behavior is modeled by a combination of rigid links and torsional springs.

In general, design of compliant mechanisms is driven by the relationship between the input and output forces and deflections. This force-deflection relationship is the focus of this study, as applied to single-piece compliant mechanisms. An optimization-based numerical method for the topological synthesis of such compliant mechanisms is presented here. This method generates the optimal structural form of a compliant mechanism for specified input force and output deflection requirements. This type of study is important because we believe that it lays a foundation for the automated design of compliant mechanisms with general specified force-deflection relationships.

The remainder of the paper is organized as follows. In the next section, we present the force-deflection design of a beam in order to show how the present design problem is different from the more common structural optimization for maximum stiffness, and also explain the basis for our multi-criteria formulation. The following section includes a statement of the synthesis problem and the formulation of the multi-criteria optimization problem. Next, the problem implementation using two design parameterizations, viz. truss ground structure and 2-D continuum is presented. Two examples, one in the 2-D domain and the other in 3-D, are described in the subsequent sections.

Design for Required Deflection of a Beam

In this section we present an example of designing a simply supported beam with a specified force-deflection relationship. The example shows a certain difficulty associated with the "design-for-deflection" problem formulation and a way to overcome the difficulty. The beam problem is a simplified model of the continuum problem in that the topology (a single open segment) and shape (a straight line) are known \( a \ priori \), and it is the size that is optimized. Consider the following design for required deflection problem for a straight beam (Fig. 2)
with the objective of minimizing the material volume (Barnett, 1961):

\[
\text{Minimize Volume} = \int_0^l A w(x) \, dx
\]

Subject to: \( \text{Equilibrium equations that give } M \) and \( m \)

\[
\Lambda: \int_0^l \frac{M_m}{E\beta w(x)} \, dx - \Delta = 0 \quad \text{(deflection constraint)}
\]

Data: \( E, \Delta, L, p(x) \) \hspace{1cm} (1)

(For the assumed rectangular cross-section, \( A = \alpha w \) and \( I = \beta w \), where \( \alpha \) and \( \beta \) are scalar quantities that depend on the chosen value for the thickness of the beam. \( \Delta \) is the desired deflection at \( x^* \), and \( M \) and \( m \) are moments due to the actual loading, \( p(x) \), and unit virtual (dummy) load applied at \( x^* \), respectively.)

Using the variational approach (Haftka and Gürdal, 1993), the optimum value of beam width can be obtained as (Ananthasuresh et al., 1994c):

\[
w^*(x) = \frac{1}{\Delta} \left( \int_0^l \frac{\alpha M_m}{E\beta} \, dx \right) \sqrt{\frac{M_m}{E\alpha\beta}}
\]

It is worth noting that when the quantity \((M_m)\) becomes less than zero anywhere in the design domain, the value of \( w^*(x) \) will be imaginary and hence there is no solution. This implies that the problem is not well-posed. To rectify this problem, we introduce a constraint on the mean compliance to make the beam problem discussed above a better posed problem. **Mean compliance** is defined as the work done by the external forces. With the addition of a constraint on the mean compliance (Eq. (3)), the optimized width can be obtained as in Equation (4):

\[
\Gamma: \int_0^l \frac{M^2}{E\beta w(x)} \, dx - C \leq 0 \quad \text{(mean compliance constraint)}
\]

\[
w^*(x) = \sqrt{\frac{(\Lambda M_m + \Gamma M^2)}{E\alpha\beta}}
\]

where the Lagrange multipliers \( \Gamma \) and \( \Lambda \), both non-negative, can be found from the two constraints. In choosing \( C \), we can gain control over \( \Gamma \) to avoid situations where we get a negative value under the square root sign. Thus, a solution is guaranteed for any load condition provided that mean compliance, \( C \), is appropriately chosen. The physical interpretation of this is that the deflection-requirement constraint alone is not sufficient and an additional constraint on the mean compliance is necessary in designing compliant structures and mechanisms. The consideration of mean compliance implies an indirect control of the overall stiffness of the structure. We extend this notion further to general 2-D and 3-D continuum situations in the form of a multi-criteria optimization problem which takes into account both the deflection-requirement and mean compliance. The general problem statement is described next.

**Problem Formulation**

The need for including both the deflection and mean compliance can also be explained from a different standpoint. When designing compliant mechanisms, one must account for both the kinematic (motion) requirements and the structural (loading) requirements simultaneously. A compliant mechanism that is designed only to be flexible can meet kinematic requirements through deformation, but may be too flexible to resist any additional loads. Furthermore, as shown in the beam example, designing for deflection (flexibility) alone can also lead to infeasible solutions. On the other hand, a compliant mechanism that is designed to be only stiff to resist loads will meet structural requirements, but may require too much force to achieve a desired motion, producing a poor mechanical advantage and high stresses. To illustrate the point further, consider a general compliant device designed to grasp some workpiece when the load, \( F_A \) is applied (Figs. 3a and 3b). Before contacting the workpiece, the device must be designed so that it is flexible and can achieve the desired motion, as in load condition 1 (Fig. 3a). After contacting the workpiece however, it must be

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**Nomenclature**

- \( F_A \) = applied force
- \( \Delta \) = required deflection
- \( w(x) \) = beam width
- \( L \) = beam length
- \( x^* \) = specified point in beam span
- \( M \) = moment due to actual loading
- \( m \) = moment due to unit virtual loading
- \( E \) = modulus of elasticity
- \( p(x) \) = distributed loading on the beam
- \( I \) = area moment of inertia
- \( \alpha, \beta \) = scalar multipliers
- \( \Lambda \) = Lagrange multiplier
- \( w^*(x) \) = optimal beam width
- \( \Gamma \) = Lagrange multiplier
- \( C \) = upper bound on compliance
- \( f_A \) = applied nodal force vector
- \( \Gamma_1 \) = location of applied force
- \( \Gamma_2 \) = location of desired output
- \( f_B \) = virtual (dummy) nodal load vector
- \( \beta \) = virtual (dummy) load
- \( u_A \) = nodal displacement vector due to \( f_A \)
- \( K_1, K_2 \) = stiffness matrices
- \( u_B \) = nodal displacement vector due to \( f_B \)
- \( u_{\text{up}} \) = nodal displacement vector due to \( f_{\text{up}} \)
- \( N \) = total number of truss members
- \( V^* \) = total volume constraint
- \( A_i \) = cross-sectional area
- \( L_i \) = length of truss member
- \( A_{\text{up}} \) = upper bound constraint
- \( A_{\text{low}} \) = lower bound constraint
- \( a, b \) = dimension of void in unit cell
- \( \theta \) = orientation of unit cell
- \( \text{conv} \) = convergence criteria
max \( f_{\Delta u} \) = max \( (v^T K_m u_A) \) = max \( (MPE) \) \( \tag{5} \)

Thus, maximizing the deflection is equivalent to achieving the required flexibility of the structure, which is equivalent to maximization of the mutual energy. This provides the problem formulation for the mechanism design problem, as shown in Eq. (6). The constraints are two equilibrium equations, one due to the applied load, \( f_A \), and another due to the dummy load, \( f_B \).

\[
\begin{align*}
\max (v^T K_m u_A) \\
\text{subject to:} \quad K_m u_A = f_A \\
K_m v_B = f_B
\end{align*}
\( \tag{6} \)

The second loading condition is now considered (Fig. 5). This part of the two-part problem is termed the “structure design”, and is where the structural requirements are met by maximizing the stiffness. Here the point \( \Gamma_1 \) is considered fixed, and the load \( F_B \) is applied at point \( \Gamma_1 \) in the opposite direction, which accounts for the resistance of the workpiece. Maximizing the stiffness is equivalent to minimizing the strain energy, \( u^T K_m u_B \), where \( u_B \) is the nodal displacement vector due to this set of loading, and \( K_m \) is the symmetric global stiffness matrix, as shown in Eq. (7). The constraint is a single equilibrium equation.

\[
\min (u^T K_m u_B) \\
\text{subject to:} \quad K_m u_B = -f_B
\( \tag{7} \)

The structure design problem is the classical structural minimization of strain energy. Much work has been done in this area by researchers in the structural optimization field, including the use of the homogenization method to perform topology optimization for structures (Bendsøe et al., 1993). These methods predict the optimal structural form in terms of rigidity and material economy. What is unique about the compliant mechanism design is the incorporation of the mutual potential energy into the optimization problem formulation. This two-part problem formulation specifically addresses the need for flexibility as well as structural rigidity when designing compliant mechanisms.

Now that the two loading conditions are modeled in terms of potential energies, they must be combined in some way. One way to do this is by using a weighted linear combination of the two objectives in a multi-criteria optimization approach (Ananthasuresh et al., 1994a, 1994b). However, often the values of two objectives differ by several orders of magnitude, depending on the problem specifications. When this happens, the larger one will dominate. This effect can be compensated for by using appropriate weighting factors, but the value of these factors varies from problem to problem. It is not possible in general to predict the appropriate weighting factors so that both objectives are considered equally in the solution. Therefore, a new method of combining the two objectives is needed.
Another way to combine the two objectives while avoiding problems due to orders of magnitude is by using a ratio. Since the mutual energy is to be maximized and the strain energy is to be minimized, the problem can be posed as a ratio of mutual to strain energy. In this formulation, no weighting factors are needed. The constraints are the equilibrium equations from each of the two loading conditions, plus additional constraints on the total material resource, and lower and upper bounds on the design variables. This formulation represents a new method of incorporating both the flexibility and stiffness requirements into a single design objective, and is discussed in detail in the next section.

**Implementation**

The multi-criteria problem formulation is implemented using finite element analysis and a numerical solution of the optimization problem. Two approaches are discussed below, one using a truss ground structure, and another using a two-dimensional continuum with homogenization.

**Truss Ground Structure.** The problem formulation is implemented first by discretizing the available design domain into finite elements. In this case, truss elements were chosen, and a full ground structure (where every node is connected to every other node by a truss element) was used. Although truss ele-
ments can only support tension and compression modes of loading, they were chosen as finite elements because of their simplicity. Clearly, incorporating bending modes of loading is important when modeling compliant mechanisms. But this can be accomplished by using a sufficient number of truss elements. For instance, a pair of truss elements can simulate a beam in bending, where one element acts as the portion of the beam in tension, and the other element acts as the portion of the beam in compression. So the mechanics of bending is accounted for indirectly by using a combination of truss elements. This notion is further illustrated in the design examples.

An algorithm based on the truss ground structure was created. In this case, the design variables are the cross-sectional areas of the truss members, \( A_i \). The formulation is shown in Eq. (8), where \( K_1 \) is the stiffness matrix for the mechanism design problem and \( K_2 \) is the stiffness matrix for the structure design problem.

\[
\begin{align*}
\max \quad & \left[ \frac{\sqrt{K_1} K_1 u_A}{u_A K_1 u_A} \right] \\
\text{subject to:} \quad & K_1 u_A = f_A \\
& K_2 u_B = -f_B \\
& \sum_{i=1}^{n} A_i L_i \leq V^* \\
& \delta_{\text{lower}} \approx A_i \approx \delta_{\text{upper}}
\end{align*}
\]

The basic computational procedure is outlined in Fig. 6. The user must specify the input force, as well as the desired output deflection in terms of its location and direction. It is important to note that the algorithm then seeks to maximize the actual output deflection in the desired direction and to minimize the mean compliance under a different loading condition. The user must also specify an initial guess for the design variables as a starting point for the algorithm, as well as a maximum step size for updating the design variables. The algorithm then employs the sequential linear programming (SLP) method for constrained optimization.

Two-Dimensional Continuum. For purposes of comparison and validation of the truss algorithm, a continuum problem formulation was also created and implemented using the homogenization method.

Homogenization theory was originally developed to solve structural optimization problems based on optimality criteria methods. The method provides an optimal topology, shape and size for structures with maximum stiffness and a prescribed material resource constraint (Bendsøe and Kikuchi, 1988; and Bendsøe et al., 1993). It divides the design domain, \( \Omega \), into a microstructure consisting of material and void, as shown in Fig. 7. Here the design variables are \( a \) and \( b \), which define the dimensions of the void in the unit cell, and \( \theta \), which defines the orientation of the cell. The homogenization method determines the optimal values for \( a \), \( b \), \( \theta \) of each cell (and thus the material distribution) based on the applied loads and constraints and a predetermined resource constraint. The result of the homogenization method is a homogenized image, which is a finite element representation of the optimal topology. The optimal material density of each element or cell is characterized in gray scale, where black represents the most dense material and white represents void.

The formulation for the two-dimensional continuum case is the same two-part problem using mutual and potential energies, and is based on variational calculus. Nishiwaki et al. (1996) discuss the computational details of the homogenization method and examine the effect of mesh size and volume constraint on the topology of the solution using this formulation. In Eq. (9), \( \Gamma_1 \) represents the boundary where the force \( f_1 \) is applied, and \( \Gamma_2 \) represents the boundary where the dummy force \( f_2 \) is applied. The constraints are the equilibrium equations plus the total volume constraint. The same multi-criteria optimization scheme is used, and is solved using SLP. In this case, the design variables are the densities of the homogenized cells. The results of both the truss and continuum methods should be topologically the same, as is evidenced in the next section.

\[
\begin{align*}
\max \quad & \left[ \frac{\text{mutual energy}}{\text{strain energy}} \right] \\
\text{subject to:} \quad & \int_{\Gamma_1} f_1 u_1 d\Gamma = \int_{\Gamma_2} (-f_2) u_2 d\Gamma \\
& \int_{\Omega} (1 - ab) d\Omega \leq V^*
\end{align*}
\]

Results

Two design examples, one in the 2-D domain and the other in 3-D, are presented which illustrate the application of the synthesis method.
Example 1: Compliant Gripper Mechanism: 2-D Truss Ground Structure. The first example is a design of a two-dimensional compliant gripper mechanism, and the results are compared to those of the continuum method. A symmetric half-view of the design problem is shown in Fig. 8a, where the dashed line represents the design domain, and the boundary conditions (nodal constraints) are as indicated. The design specifications are that an applied force, $F_A$, cause the motion, $\Delta$, in the vertical direction. The direction of the desired output displacement is specified in terms of a unit vector. Notice that the point of application of the applied force is constrained to move only in the horizontal direction due to the symmetry of the problem. The other problem specifications are indicated in Fig. 8a. The initial guess shown in Fig. 8b is a full ground structure with cross-sectional areas equal to 0.03 m².

In this case, the algorithm converged after 448 iterations. The optimal solution is shown in Figure 8c, which is illustrated as a gray scale plot, where the truss members whose design variable reached (or was close to) the lower bound constraint are not shown. A finite element analysis of this solution was performed in order to verify the solution behavior. The result is shown in Fig. 8d, where the undeformed shape is denoted by the dashed lines and the deformed shape is denoted by the solid lines. The displacements calculated here are for the optimal design subject to the applied load, $F_A$. The direction of the actual displacement at the point of interest is not in exact agreement with the problem specification. This is because the problem formulation requires only that we maximize the output deflection in a specified direction, which roughly means maximizing the individual deflection components in the x and y directions, but does not control the ratio of the x and y deflection components.
The design objective as well as the mutual potential energy and strain energy are plotted as a function of the algorithm iteration number in Fig. 8e. Notice that the mutual potential energy is maximized and the strain energy is minimized, which indicates that the algorithm is working correctly. (The values of mutual energy and strain energy are scaled for the purposes of plotting.)

Once the mechanism topology is determined, an actual compliant mechanism prototype can be designed. Notice that for this example, the interior area of the gripper, area QRUTS in Fig. 8c, acts as a structure. Therefore in designing the prototype part for manufacture, this area was filled in with material. Members PQ and PR together act as a beam, which implies that a single segment is required to act in bending. A compliant gripper design was created based on this idea, and was created using layered manufacturing on the Stratasys 3D Modeler (1993). The gripper is shown in Fig. 8f, along with a 3-D version. The 3-D version was designed by rotating the 2-D design by 120 degrees about the center axis.

Example 1: Compliant Gripper Mechanism: 2-D Continuum. This gripper design problem was also solved using the two-dimensional continuum homogenization method. The problem specifications are described in Fig. 9a. In this case a 60 x 30 mesh was prescribed. Notice that the applied load is given in terms of a unit load, and that the problem specifications are non-dimensionalized parameters because their scalar values do not affect the topology of the optimized solution. The initial guess is a uniform distribution of cell densities, with the volume constraint as 20 percent of this initial volume.

The result of the continuum method is shown in Fig. 9b. Notice the agreement between this topology and the topology of the truss method solution shown in Fig. 8c. Figure 9c shows a finite element model of this solution in its deformed shape, along with the direction of the actual output displacement. Similar to the result of the truss method, the direction of the actual output displacement is not in exact agreement with the problem specification. This disagreement can be explained as follows. Recall that the mechanism part of the formulation requires maximum output deflection in the specified direction. This requirement does not guarantee that the direction of the actual output deflection will be in the specified direction, only as close as it can be given the constraints of the problem. This example illustrates the fact that it is not always possible to obtain good agreement between the specified direction of the output deflection and the actual direction. The direction and magnitude of the output deflection can be further refined, however, with an additional constraint in shape and size optimization, and through material selection. The algorithm converged after 309 iterations, and the convergence history is shown in Fig. 9d (the strain energy is scaled for purposes of plotting).

Example 2: Three-Dimensional Compliant Mechanism: 3-D Truss Ground Structure. The second design example illustrates the capability of this synthesis method to handle three-dimensional problems. As shown in Fig. 10a, the design domain is a cube indicated by the dashed lines, where the nodes at each corner of the bottom face are fixed. The input force, \( F_x \), is applied at the center of the bottom face of the cube in the \( x \)-direction. The direction of the desired output displacement, \( \Delta \), is in the positive \( x \)-direction at the indicated location, and the other problem specifications are described as well. The initial guess is a cubic full ground structure, as shown in Fig. 10b.

In this case, the algorithm converged after 33 iterations. The optimal solution is shown in the gray scale plot of Fig. 10c, where the truss members whose design variable reached (or was close to) the lower bound constraint are not shown. A finite element model of the solution is shown in Fig. 10d, where the undeformed shape is denoted by the dashed lines and the deformed shape is denoted by the solid lines. The displacements calculated here are for the optimal design subject to the applied load, \( F_x \). The direction of the actual output displacement in this case is in close agreement with the input specification. The design objective as well as the mutual potential energy and strain energy are plotted as a function of the algorithm iteration number in Fig. 10e. The mutual potential energy is maximized and the strain energy is minimized, which once again indicates that the algorithm is working correctly. (The values of strain energy and the objective function are scaled for the purposes of plotting.) A 3-D compliant mechanism was built using nylon and the Stratasys 3D Modeler (1993), and is shown in Fig. 10f. This device can be used as an internal gripper to grasp hollow objects.

Conclusions

An automated method for the topological synthesis of compliant mechanisms is developed for a specific class of design problems, where the device must be both flexible and stiff for a given task. The problem formulation presented here is a new way of combining these conflicting design objectives using multi-criteria optimization. These compliant mechanism solutions are inherently different from the stiffest structure solutions obtained by minimization of strain energy because of the inclusion of the mutual potential energy in the problem formulation. The "design for required deflection" beam example provides the motivation for this new two-part problem formulation.

As the design examples illustrate, it is possible to generate compliant mechanism topologies using a truss ground structure for a given set of input force and output deflection requirements. The results of this method agree topologically with those produced by the two-dimensional continuum/homogenization method, indicating that the mechanics of bending is accurately modeled using a combination of truss elements. Moreover, the functionalisity of the solutions is evidenced by both the finite element models and the prototype designs.

The focus of this work is on design for required deflection, where both the kinematic and structural requirements are considered. The issue of mechanical advantage has not been directly addressed, however. This is certainly an important aspect of mechanism design, and future efforts will be directed at developing a problem formulation which incorporates mechanical advantage. Future efforts will also be directed at incorporation of shape and size optimization to perform dimensional synthesis required to quantify the final compliant mechanism design, as well as addressing material considerations.

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