Introduction to the J-integral

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The purpose of this lecture is to briefly introduce the J-integral, which is widely used in fracture mechanics. To the extent possible, we shall keep the lecture self-contained. This will mean a repetition of familiar concepts for some students, but we hope that the review will be beneficial overall. Ideally, we would like to see lots of applications/examples of the concept to become familiar with it. We will not be able to do so in class today but this can be done through homeworks. One of the principal objectives of this lecture is to present a unification of concepts being learnt in concurrent courses. The material covered will serve as a useful point of departure when we subsequently start learning about numerical methods to simulate fracture. Above all, the J-integral is a central aspect of (nonlinear) fracture mechanics and it behooves us to understand its origin instead of accepting it as a given. Such understanding is essential to recognize and conceive of its applications when solving new problems.

1 What is the J-integral

Consider a two-dimensional elastic body as shown in Figure 1 and let $\Gamma$ be a closed contour that encloses a simply connected region. The J-integral is defined

1. Figure 1: Coordinate system and normal used in defining the J-integral in (1).
as

\[ J \triangleq \int_{\Gamma} \left( W dx_2 - t \cdot \frac{\partial \mathbf{u}}{\partial x_1} \right) ds, \]  

(1)

where \( W \) is the strain energy density, \( t \) is the traction, \( \mathbf{u} \) represents the displacement and we have chosen the usual coordinate system \((x_1, x_2)\). It is fair to ask what significance, if any, does the quantity defined in (1) have. The following discussions will help answer this question to some extent.

### 1.1 Elastostatic equilibrium, no singularities \[ \Rightarrow J = 0 \]

We claim that in the absence of any sources of singularities (such as cracks) in the area enclosed by \( \Gamma \), the value of \( J \) equals zero irrespective of the choice of the contour \( \Gamma \). Let us demonstrate this. Observe that we can rewrite (1) as

\[ J = \int_{\Gamma} \left( W n_1 - (\sigma \cdot n) \right) \cdot \frac{\partial \mathbf{u}}{\partial x_1} ds, \]  

(2)

where \( n = (n_1, n_2) \) is the unit outward normal to \( \Gamma \) and \( \sigma \) is the Cauchy stress. By appealing to the divergence theorem, the first term in (2) simplifies as

\[ \int_{\Gamma} W n_1 ds = \int_{\Gamma} (W \mathbf{e}_1) \cdot n ds = \int_{\Omega} \text{div}(W \mathbf{e}_1) d\Omega = \int_{\Omega} \frac{\partial W}{\partial x_1} d\Omega. \]

By the chain rule,

\[ \frac{\partial W}{\partial x_1} = \frac{\partial W}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x_1} = \sigma_{ij} \frac{\partial^2 u_i}{\partial x_j \partial x_1}, \]

so that we get

\[ \int_{\Omega} W n_1 ds = \int_{\Omega} \sigma_{ij} u_{1,i,j} d\Omega. \]  

(3)

The second term in (2) simplifies similarly:

\[ \int_{\Gamma} (\sigma \cdot n) \cdot \frac{\partial \mathbf{u}}{\partial x_1} ds = \int_{\Gamma} (u_{i,1} \sigma_{ij} \mathbf{e}_j) \cdot n ds = \int_{\Omega} \text{div}(u_{i,1} \sigma_{ij} \mathbf{e}_j) d\Omega \]

\[ = \int_{\Omega} (u_{i,1} \sigma_{ij})_{,j} d\Omega \]

\[ = \int_{\Omega} (u_{i,1} \sigma_{ij} + u_{i,1} \sigma_{ij,j}) d\Omega \]

\[ = \int_{\Omega} u_{i,1,j} \sigma_{ij} d\Omega, \]  

(4)

From (3) and (4), we get conclude that

\[ J = \int_{\Gamma} \left( W n_1 - (\sigma \cdot n) \right) \cdot \frac{\partial \mathbf{u}}{\partial x_1} ds = \int_{\Omega} (\sigma_{ij} u_{1,i,j} - u_{i,1,j} \sigma_{ij}) d\Omega = 0. \]  

(5)
Questions: In the above arguments,

(i) where did we use the fact that the body was elastic?

(ii) where did we use the assumption that the body was in static equilibrium?

(iii) where did we use the assumption that there were no singularities such as cracks?

1.2 J for a cracked body at static equilibrium

When the contour \( \Gamma \) encloses a region where the solution fields have singularities, such as in the presence of cracks, the J-integral may not vanish. However, a useful property still holds. Referring to figure 2, this time we consider any two paths \( \Gamma_1 \) and \( \Gamma_2 \) that start and end on opposite faces of the crack. Notice that \( \Gamma_1 \) and \( \Gamma_2 \) are not closed since they are disconnected at the crack faces. Let us examine the values of the J-integral evaluated over these contours.

To this end, first we construct a closed contour \( \Gamma = \Gamma_1 \cup \Gamma_+ \cup \Gamma_2 \cup \Gamma_- \) as indicated in figure 2. Notice that \( \Gamma \) is now a closed curve enclosing a region with no singularities. Therefore by our previous arguments, we know \( J_\Gamma = 0 \). Splitting the contour integral over \( \Gamma \) into contributions from each of its four subsets, we get

\[
J_\Gamma = J_{\Gamma_1} + J_{\Gamma_+} - J_{\Gamma_-} - J_{\Gamma_2} = 0
\]  

Let us choose the coordinate system \( (x_1, x_2) \) in alignment with the crack as shown in figure 2. Notice that along the curves \( \Gamma_{\pm} \), we have \( t = 0 \) and \( n_1 = 0 \).
Therefore
\[ J_{\Gamma_\pm} = \int_{\Gamma_\pm} (Wn_1 - t \cdot u_1) \, ds = 0. \]  
(7)

Eq. (6) hence shows that \( J_{\Gamma_1} - J_{\Gamma_2} = 0 \), i.e., that \( J_{\Gamma_1} = J_{\Gamma_2} \). This property of \( J \) evaluated along contours emanating on one crack face and terminating on the other, is frequently referred to as the path-independence of property.

**Questions:** Examine the above discussion carefully.

(i) What is the reason behind the signs appearing in (6)?

(ii) Consider the contour \( \Gamma_1 \) shown in figure 2. Will the sign of \( J_{\Gamma_1} \) change if we traverse \( \Gamma_1 \) clockwise instead of counter clockwise?

(iii) In demonstrating the path independence property, did we explicitly use the form of the singularity of the crack tip fields?

(iv) Similarly, did we assume anything about the shape of the crack? Does it have to be straight or can it be curved? Does the crack tip have to be sharp or can we also permit notches?

Although we have shown that the value of \( J \) is independent of the path, we do not know its value. In the context of LEFM, it can in fact be shown that \( J = G \), the strain energy release rate! The computation is quite straightforward but occasionally tedious. The simpler case of a mode III crack may help convince you of this fact.

**Questions:** The mode III displacement field around a crack tip is given by
\[ u_3(x, y) = \frac{1}{\mu} \text{Im} \left( \frac{2K_{III}}{\sqrt{2\pi}} \sqrt{x + iy} \right). \]

From the G-K relationship, we know \( G = K_{III}^2/2\mu \). We would like to verify our claim that \( J = G \) using the known displacement field around the crack. To this end, choose \( \Gamma \) to be a circular arc radius \( r \) and centered at the crack tip. Working in polar coordinates will help.

(i) What should be the angular coordinates of the end points of \( \Gamma \)?

(ii) Write down the unit outward normal to \( \Gamma \) at a generic point \((r, \theta)\).

(iii) Compute the value of the strain energy density \( W \) at \((r, \theta)\).

(iv) Compute the necessary derivatives of displacements, the strain and then the stress.
Conclude that $J = K_{III}^2/2\mu$

**Solution:** Although it is convenient to work in a polar coordinate system, notice that our definition of $J$ explicitly refers to a Cartesian coordinate system at the crack tip. This will be a minor hassle and requires attention. From the given expression for the displacement, we have

$$u_3(r, \theta) = \frac{1}{\mu} \text{Im} \left( \frac{2K_{III}}{\sqrt{2\pi}} \sqrt{r}e^{i\theta} \right) = \frac{2K_{III}}{\sqrt{2\pi\mu}} \sqrt{r} \sin \frac{\theta}{2}. \tag{8}$$

The only non-zero components of stress and strain are $\sigma_{31}, \sigma_{32}$ and $\varepsilon_{31}, \varepsilon_{32}$ respectively. It is straightforward to verify that

$$\sigma_{32} + i \sigma_{31} = \frac{K_{III}}{\sqrt{2\pi z}}. \tag{9}$$

Our objective then is to compute

$$J = \int_{\theta = -\pi}^{\pi} \left( W_{n1} - \sigma_{k3} \frac{\partial u_3}{\partial x_1} n_k \right) r d\theta. \tag{10}$$

Let us evaluate the strain energy density:

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} (\sigma_{31} \varepsilon_{31} + \sigma_{32} \varepsilon_{32})$$

$$= \frac{1}{2\mu} (\sigma^2_{31} + \sigma^2_{32})$$

$$= \frac{|\sigma_{32} + i \sigma_{31}|^2}{2\mu}$$

$$= \frac{K_{III}^2}{4\mu \pi |z|}. \tag{11}$$

The unit normal to $\Gamma$ has components $(n_1, n_2) = (\cos \theta, \sin \theta)$. Therefore,

$$\int_{\theta = -\pi}^{\pi} W_{n1} d\Gamma = \int_{\theta = -\pi}^{\pi} \frac{K_{III}^2}{4\mu \pi r} \cos \theta r d\theta = 0.$$

The first term in (10) hence makes no contribution to $J$. Towards computing the remaining terms, we have

$$\frac{\partial u_3}{\partial x_1} = 2\varepsilon_{31} = \frac{\sigma_{31}}{\mu} = \frac{1}{\mu} \text{Im} \left( \sigma_{32} + i \sigma_{31} \right) = \frac{1}{\mu} \text{Im} \left( \frac{K_{III}}{\sqrt{2\pi z}} \right) = -\frac{K_{III}}{\mu \sqrt{2\pi r}} \sin \frac{\theta}{2}$$

Next, we have

$$\sigma_{k3} n_k = \sigma_{3r} = \mu \frac{\partial u_3}{\partial r} = \frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}. \tag{12}$$
We can now compute $J$:

\[
J = -\int_{\theta=-\pi}^{\pi} \sigma_{k3} \frac{\partial u_3}{\partial x_1} n_k r d\theta
\]

\[
= \int_{\theta=-\pi}^{\pi} \left( \frac{K_{III}}{\mu \sqrt{2\pi r}} \sin \frac{\theta}{2} \right) \left( \frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \right) r d\theta
\]

\[
= \frac{K_{III}^2}{2\pi \mu} \int_{\theta=-\pi}^{\pi} \sin^2 \frac{\theta}{2} d\theta = \frac{K_{III}^2}{4\pi \mu} \int_{\theta=-\pi}^{\pi} (1 - \cos \theta) d\theta
\]

\[
= \frac{K_{III}^2}{2\mu},
\]

which coincides with $G$ for a pure mode III crack and is exactly the result we wanted to prove. The cases of mode I and II cracks can similarly be verified.

While it is reassuring to see that $J$ is a familiar quantity in LEFM, we also realize that it does not add any new information. Indeed, some of the main applications of $J$ are in nonlinear fracture, and in particular, for studying fracture of nonlinearly elastic and monotonically loaded elasto-plastic materials. This is a topic by itself and can be taken up in future lectures depending on the interest of the class. We will not pursue such applications of $J$ any further at this point.

## 2 J-integral as a conservation law

Where did the J-integral come from? The proposed definition (1) appears to be the work of a wizard, which is quite unsettling. Without providing all the necessary details, we state here a useful result called Noether’s theorem that relates symmetries (coordinate tranformations) with conservation laws for certain variational systems. We will first state the result in a simplified form, understand the statement, and see how it yields the J-integral in one stroke. At the very least, this exercise should provide some understanding on the origin of the J-integral as well as motivate the fact that there may be (and are) more such integrals in existence.

A version of Noether’s theorem: Let an elastic body occupy region $\Omega$. Let $u$ denote an admissible displacement field, $\nabla u$ its gradient, and let the strain energy density be of the form $W(x) \triangleq W(x, u, \nabla u)$ so that the total strain energy of the body is given as

\[
\Phi(u) = \int_{\Omega} W(x, u, \nabla u) d\Omega.
\]

(13)
Consider a smooth family of coordinate transformations
\[ \xi(x, \eta) = x + \eta f(x) \]  
and a family of vector transformations
\[ h(u, \eta) = u + \eta g(u) \]
each parameterized by the scalar \( \eta \). If the one-parameter family of functionals
\[ \Phi_{\eta}(w) = \int_{\Omega_\eta} W(\xi, h, \nabla \xi h) d\Omega_\eta. \]
is such that \( \Phi_{\eta}(v) = \Phi(v) \) for all \( \eta \) and admissible \( v \), and if \( u \) is a stationary point of the functional \( \Phi \), then
\[ \int_{\Gamma} \left( W f_j + (g_k - f_{\ell u_k,\ell}) \frac{\partial W}{\partial u_{k,j}} \right) n_j d\Gamma = 0 \]
for every closed surface \( \Gamma \) bounding a regular subregion of \( \Omega \) and \( u \) being the unit outward normal to \( \Gamma \).

The statement of the theorem makes two important assumptions. First, that \( u \) is a stationary point of the energy functional \( \Phi \) and second, that there are certain transformations of \( x \) and \( u \) that leave \( \Phi \) invariant in some sense. Notice that coordinate change \( x \to \xi \) and the transformation \( v \to h \) are possibly different mappings for each value of \( \eta \). With these assumptions, we get that a certain integral depending on the solution \( u \) vanishes over any arbitrary closed surface \( \Gamma \).

In the next section, we will examine what it means for \( u \) to be a stationary point, at least within the context of linear elasticity. For the moment, let us see how we can recover the J-integral for a linearly elastic material from the above result. For such a material, we know
\[ W(x, v, \nabla v) = \frac{1}{2} \nabla v : C : \nabla v. \]
Consider the coordinate transformation
\[ \xi = x + \eta e_i, \quad \text{for a fixed } i, \]
which is just a translation along the direction \( e_i \), and the identity vector transformation \( h(v) = v \). We see that
\[ \nabla \xi h(u) = \nabla u, \]
and hence
\[ W(\xi, h(v), \nabla \xi h(v)) = W(x, v, \nabla v). \]
Furthermore, the Jacobian of the transformation $x \rightarrow \xi$ equals 1. Hence we trivially get that
$$
\Phi_\eta(v) = \Phi(v) = \frac{1}{2} \int_\Omega \nabla \mathbf{v} : \mathbf{C} : \nabla \mathbf{v} \, d\Omega.
$$

We have thus found one-parameter families of coordinate and vector transformations that leave $\Phi$ invariant. Noether’s theorem then tells us that a certain integral quantity equals zero. Let us see what it is. In the notation of the statement, we have
\begin{align*}
f(x) = e_i \Rightarrow f_j(x) &= \delta_{ij}, \\
g(v) = 0 \Rightarrow g_k = f_\ell u_{k,\ell} = -\delta_{i\ell} u_{k,\ell} = -u_{k,i},
\end{align*}
so that
\begin{align*}
\int_\Gamma \left( W f_j + (g_k - f_\ell u_{k,\ell}) \frac{\partial W}{\partial u_{k,j}} \right) n_j \, d\Gamma &= \int_\Gamma (W \delta_{ij} - u_{k,i} \sigma_{k,j}) n_j \, d\Gamma \\
&= \int_\Gamma \left( W n_i - \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \right) \, d\Gamma = 0,
\end{align*}
which coincides with out definition of the J-integral for the case $i = 1$. Hence we have demonstrated to ourselves that:

(i) the conservation property of the J-integral is a natural consequence of a certain kind of symmetry inherent in elasticity,

(ii) there is not just one J-integral, but in fact one for each component $i = 1, 2, 3$. Hence we can consider a vector valued J-integral, of which the one we have discussed so far corresponds to one component.

(iii) the conservation property holds not just over contour integrals in two-dimensions, but also over surfaces in three-dimensions. This provides a helpful way of computing stress intensity factors along the propagating edge of a three dimensional crack surface.

Observe that we used a very simple coordinate transformation, namely a rigid body translation along $e_i$ to arrive at what appears to be a nontrivial conservation law. It definitely seems plausible that if we can find more such symmetry transformation, we may be able to discover new conservation laws with deeper consequences. You are strongly encouraged to find out about L and M integrals, which are conservation laws with important applications in micro-mechanics with defects, and can be derived in this way.

**Question:**
(i) The conservation property of the J-integral holds also for nonlinearly elastic solids, with appropriate stress and strain measures. Indeed, notice that there is no mention of linear elasticity in the statement of the theorem. In our demonstration of J as a consequence of Noether’s theorem however, we made essential use of the assumption that the material was linearly elastic. Where?

(ii) Did we assume anything about the symmetry of the material? For example, did we assume that the material was isotropic?

(iii) Did we assume that the material distribution in the body was homogeneous? That is, can $C = C(x)$ or did we implicitly assume $C = \text{a constant}$?

3 Linearized elastostatics from a variational principle

Noether’s theorem applies to systems that follow a variational principle. In the current context, this means that the equations of elastostatic equilibrium be derivable as the minimization of an energy functional. We demonstrate here that the equilibrium equations for a linearly elastic solid can indeed by derived this way. Equilibrium equations for bodies composed of hyper-elastic materials can also be derived this way, but we omit this case to avoid introducing new stress/strain measures and concepts from continuum mechanics that may not be familiar to all the students in the class.

In the following, we consider a linearly elastic body $\Omega$ composed of a material with constitutive relation $\sigma = C : \varepsilon$. Under imposed displacements $u_0$ along the boundary $\partial \Omega$, we would like to compute the response of the body, i.e., its displacement field $u$.

3.1 Admissible displacements

Clearly, we need to write down the equations of static equilibrium and solve for $u$. This can be achieved with free body diagrams and invoking Newton’s second law. Here, we follow a different idea, namely that of strain energy minimization. Recall that the strain energy corresponding to a displacement field $v$ is given by

$$\Phi(v) = \int_\Omega W(\varepsilon[v]) \, d\Omega,$$  \hspace{1cm} (21)
where $\varepsilon[v] = v_{(i,j)}$ is the strain associated with $v$, and the strain energy density $W$ is defined as

$$W(\varepsilon) = \frac{1}{2} \sigma : \varepsilon = \frac{1}{2} \varepsilon : C : \varepsilon. \quad (22)$$

Observe that (21) defines $\Phi$ as functional, i.e., $\Phi$ assigns a scalar $\Phi(v)$ to a displacement field $v$. We follow the principle that the actual displacement field $u$ of the body is one that in fact minimizes the strain energy $\Phi$. How can we search for such a displacement? Well, to claim that $u$ is a solution, we need to compare $\Phi(u)$ against the values of $\Phi$ realized over all other possible displacements $v$. In performing this comparison, we shall restrict our attention to displacement fields $v$ that satisfy the prescribed boundary conditions. Additionally, we shall require that all candidate displacements $v$ be sufficiently smooth so that all the derivatives appearing in our computations will make sense. We call such candidate displacement fields as the set of admissible solutions. Hence our objective is to find $u$ such that $\Phi(u) \leq \Phi(v)$ for all admissible solutions $v$.

### 3.2 Static equilibrium: Euler-Lagrange equations

We approach the minimization problem with tools from calculus. We shall request that $u$ be a stationary point of the functional $\Phi$ by insisting that

$$\langle \delta \Phi(u), w \rangle \triangleq \left. \frac{d}{d\eta} \Phi(u + \eta w) \right|_{\eta=0} = 0, \quad (23)$$

for all smooth vector fields $w$ satisfying $w = 0$ along the boundary of $\Omega$. Let us examine (23) closely. First, notice that any admissible displacement $v$ can be expressed in the form $v = u + \eta w$ with $w$ satisfying homogeneous Dirichlet bc's. Hence we are sampling the functional $\Phi$ along all possible perturbations of the solution $u$. The derivative condition should be very reminiscent of the first derivative test in calculus. The intuition behind it is indeed the same—$\Phi$ has a stationary point at $u$ if its directional derivative along all admissible perturbations is zero. Make sure you understand (23) or consult your peers who have seen this material before.

We understand (23) to be a necessary condition for $u$ to be a solution of the
linear elasticity problem. Let us simplify (23). We have

\[
0 = \langle \delta \Phi(u), w \rangle = \frac{d}{d\eta} \left( \frac{1}{2} \int_\Omega \varepsilon[u + \eta w] : C : \varepsilon[u + \eta w] \, d\Omega \right) \bigg|_{\eta=0} \tag{24a}
\]

\[
= \frac{d}{d\eta} \left( \frac{1}{2} \int_\Omega (u_{i,j} + \eta w_{i,j}) C_{ijkl} (u_{k,l} + \eta w_{k,l}) \, d\Omega \right) \bigg|_{\eta=0} \tag{24b}
\]

\[
= \frac{1}{2} \int_\Omega (w_{i,j} C_{ijkl} u_{k,l} + u_{i,j} C_{ijkl} w_{k,l}) \, d\Omega \tag{24c}
\]

\[
= \int_\Omega u_{i,j} C_{ijkl} w_{k,l} \, d\Omega \tag{24d}
\]

\[
= \int_\Omega \sigma_{k\ell} w_{k,\ell} \, d\Omega \tag{24e}
\]

\[
= \int_\Omega \left( \sigma_{k\ell} w_k - \sigma_{k\ell,l} w_k \right) \, d\Omega \tag{24f}
\]

\[
= \int_\Gamma w \cdot (\mathbf{\sigma} n) \, d\Gamma - \int_\Omega w \cdot \text{div} \mathbf{\sigma} \, d\Omega \tag{24g}
\]

\[
= - \int_\Omega w \cdot \text{div} \mathbf{\sigma} \, d\Omega. \tag{24h}
\]

In this way, we have arrived at a condition that for \( u \) to be a stationary point of \( \Phi \), we should have

\[
0 = \langle \delta \Phi(u), w \rangle = \int_\Omega w \cdot \text{div} \mathbf{\sigma} \, d\Omega \quad \forall w \text{ s.t. } w|_\Gamma = 0. \tag{25}
\]

Notice that (25) imposes a tremendous constraint on \( u \), because we are completely free to choose any smooth \( w \) that satisfies homogeneous Dirichlet bcs. A well known result called the fundamental lemma of the calculus of variations in fact helps us conclude that the only way to satisfy (25) is if the integrand is itself zero, i.e., \( \text{div} \mathbf{\sigma} = 0 \).

Hence we have found an alternate way of deriving the equilibrium equations for linear elasticity without ever drawing a free body diagram. The form of the equilibrium equation remains unchanged for nonlinear materials as well; the essential arguments remain the same as what we have discussed above. But to avoid introducing new concepts from continuum mechanics, we have intentionally omitted the general case.

Questions:

(i) Justify the computation from (24a) to (24b).

(ii) Justify the computation from (24b) to (24c). In particular, evaluate the derivatives carefully.

(iii) What properties of \( C \) were used in going from (24c) to (24d).
(iv) Justify the manipulation from (24e) to (24f).

(v) How did we switch from a domain to boundary integral in (24f) → (24g)?

(vi) Why does the boundary integral in (24g) vanish?

(vii) Justify the fundamental lemma of calculus of variations in one dimension. That is, suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is a smooth function and $\int_0^1 f(x)g(x)dx = 0$ for all smooth functions $g : [0, 1] \rightarrow \mathbb{R}$ with $g(0) = g(1) = 0$. Show that $f(x) = 0$ for all $x \in (0, 1)$. Looking this up on Wikipedia is OK.