Analytical solution of the modified Reynolds equation for squeeze film damping in perforated MEMS structures

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Abstract

The squeeze-film damping in perforated structures is modelled using a modified Reynolds equation that includes compressibility and rarefaction effect. This equation is linearized and transformed to the standard two-dimensional diffusion equation using a simple mapping function. The analytical solution is then obtained using Green’s function. The solution thus obtained adds an additional term $\Gamma_1$ to the damping and spring force expressions derived by Blech for compressible squeeze flow through non-perforated plates. This additional term contains several parameters related to perforations and rarefaction. Setting $\Gamma_1 = 0$, one recovers Blech’s formulae. We compute the squeeze film forces using these new formulae and compare the computed forces with the solution of 3D Navier–Stokes equation solved using ANSYS for different perforation ratios (ratio of hole to cell dimensions). The results match very well. The approximate limit of maximum frequencies under which the formulae give reasonable results is also discussed. Although the main result is derived for a rigid plate under transverse motion, we discuss the effect of flexibility of the structure by deriving results for a flexible plate under a specified set of boundary conditions and comparing the results with that of a suitably modified rigid plate result. For small amplitude motion, the results show that a suitably modified rigid plate model can capture the effect of flexibility through a simple scaling factor.

Keywords: Squeeze-film damping; Reynolds equation; Perforations; Analytical results

1. Introduction

The dynamic response of MEMS devices operating in air, air damping plays an important role. Air-damping, such as damping due to squeeze-film flow and Couette flow, is present in micro-devices executing different types of motion such as transverse motion, parallel motion, etc., of a movable rigid or flexible plate with respect to another fixed plate. The squeeze-film damping which dominates over other losses [1] plays a significant role in the performance of many devices such as accelerometers, gyroscopes and torsional mirrors. Squeeze-film damping has been extensively analyzed for different cases such as vibration of parallel rigid plates [2–4] or flexible plate [5] with thin to ultra-thin gaps and from smooth to rough surfaces [4,6]. With the increasing application of perforated structures in MEMS devices for reducing squeeze film damping [3], for efficient etching [7], or for controlling stiffness of the structure [8], perforations and their effects on dynamic characteristics need to be modelled carefully.

To include perforation effects in squeeze-film damping calculations, several methods have been proposed. In most analyses, a repetitive pattern of holes are considered on the plate with each hole having its own domain of influence. This individual domain of influence is called a cell. For incompressible flow, Škvor [9] modelled squeeze film damping under each cell by taking ambient pressure boundary condition on the hole rim (i.e., neglecting flow through the hole) and zero flow rate across each cell. Kim et al. [10] have performed experiment on different perforated models and registered some deviation from the theoretically calculated damping force. They have mentioned the zero pressure boundary condition on the hole rim (i.e., neglecting flow through the hole) and zero flow rate across each cell. Kim et al. [10] have performed experiment on different perforated models and registered some deviation from the theoretically calculated damping force. They have mentioned the zero pressure boundary condition on the hole rim (i.e., neglecting flow through the hole) and zero flow rate across each cell. Kim et al. [10] have performed experiment on different perforated models and registered some deviation from the theoretically calculated damping force. They have mentioned the zero pressure boundary condition on the hole rim (i.e., neglecting flow through the hole) and zero flow rate across each cell. Kim et al. [10] have performed experiment on different perforated models and registered some deviation from the theoretically calculated damping force. They have mentioned the zero pressure boundary condition on the hole rim (i.e., neglecting flow through the hole) and zero flow rate across each cell.
system with circular and oval holes, Homentcovski and Miles [11] have calculated fluid damping as the sum of squeeze film damping and the loss through the holes. In order to couple the effect of holes and the squeeze-film damping, Bao et al. [16] have derived a modified Reynolds equation under the assumption of incompressible flow for perforated system operating at low frequencies. They have also validated their results numerically using ANSYS for 2D case only and experimentally with the results presented by Kim et al. [10].

Veijola et al. [12,13] have modelled the perforation effect by calculating the equivalent electrical impedance for squeeze film damping, flow through the holes, end effect of holes, and the effect of compressibility, and then by carrying out circuit simulations of the whole system using APLAC—a circuit simulation and design tool. However, they have taken perforation effect while modelling but their formula for squeeze film damping gives good results for small perforations. In another study aimed at modelling perforations in complex devices, Schrag and Wachtuka [14] have proposed a mixed level scheme in which compact models that account for the flow through the holes and the edge effects are incorporated in a finite network (FN) model of Reynolds equation. Then, the discretized FN model is solved using a standard system simulator (e.g. [15]).

Since the operating frequencies of some devices such as accelerometers, gyroscopes, etc., are considerably high and also vary over a considerable range of frequencies, the formulae derived under the assumptions of incompressible flow are not valid for such devices, especially when they operate at higher frequencies. The methods available for analyzing high frequency flows incorporating effect of holes are very time consuming as explained in Refs. [12,14]. They also require special simulators to perform simulations. So, it is desirable to develop a compact analytical formula for damping and spring force for perforated systems that can be used more easily in design optimization. In this paper, we model the perforation effect on squeeze film damping using a modified Reynolds equation in the same way as proposed by Bao et al. [16] but we include the compressibility effect, rarefaction effect and define a perforation parameter which couples the perforation dimensions with the plate dimension. Here, we neglect inertial effect which is inevitable at sufficiently high frequencies, and hence, we mention the maximum frequency range under which the present formulation is valid without any substantial error. Since the resulting equation is considerably complex and not easily solvable analytically even after normalization, we first transform the linearized form of the modified Reynolds equation to the standard 2D diffusion equation and then solve it using Green’s functions. We then derive the expressions for the squeeze-film damping force and spring force acting on perforated as well as non-perforated back plate. In this paper, we present closed form solution of squeeze-film damping and spring forces for transverse motion of rigid as well as flexible plate.

In order to check the validity of the formulae for varying degree of perforation ratio, we compare the results with 3D Navier–Stokes solution obtained using ANSYS. Here, the perforation ratio is defined as the ratio of hole to cell radius. We also show that the derived formula reduces to Blech [2] formula for non-perforated case. Finally, we remark that the formulae developed for damping and spring force due to squeeze flow can be used for perforated as well as non-perforated structures and for incompressible to medium compressible flows where inertial effects are negligible.

2. Governing equations

We consider a rectangular plate of length L along x-axis, width W along y-axis, and thickness Tp along z-axis, as shown in Fig. 1(a) and (c). The plate has $M \times N$ uniformly distributed square holes of size $L_h$ with pitch $q$ along both x and y-directions. Here, N is the number of holes in any one of the rows and M is that in any one of the columns. So, the total number of holes is $MN = M \times N$. The equivalent hole radius $b$ is calculated by comparing the resistance through a circular pipe and square channel of the same length, while the cell radius is calculated by equating the area. The expression for the equivalent hole radius $b$ and the cell radius $a$ are given by $1.096L_h/2$ and $q/\sqrt{\pi}$, respectively [13] (see Fig. 1(b)). The hole length is equal to the plate thickness, $T_p$.

When the moving plate oscillates with a velocity $V_z$ with respect to the fixed plate, the air under the plate tries to move from its position. We consider the dynamics of air flow through a typical cell shown in Fig. 1(d). The figure shows that when the plate oscillates at a certain frequency, the air volume, trapped under the annular cell, gets compressed under the cell and a part of it flows through the hole. Therefore, the region under the cell is divided into two parts, A and B. Region A contains mixed amount of air consisting of the part that gets compressed under the cell and causes the spring effect, and the part that flows to region B and contributes to the damping effect. The fluid in region B flows through the hole due to pressure difference between the two ends of the hole. So, the total pressure difference across the hole is calculated by considering the end effects in parts I and III, and the pressure difference required to maintain the flow along part II (see Fig. 1(d)). In this paper, we only consider resistance through the holes and that due to the sudden expansion or contraction of the flow at the ends of the hole and through the hole. Because, other losses such as the turning of horizontal flow under the cell to vertical flow in the hole are small as compared to the losses considered in the paper [13]. Here, we consider the flow through the holes to be incompressible, inertialess and fully developed.

The non-linear modified Reynolds equation governing squeeze-film flow that includes perforation, compressibility and rarefaction effects can be obtained by following the procedure mentioned in Ref. [16]. Thus, we get

$$
\frac{\partial}{\partial x} \left( \frac{\rho b^3 Q_{ch} \frac{\partial p}{\partial x}}{12 \mu} \right) + \frac{\partial}{\partial y} \left( \frac{\rho b^3 Q_{ch} \frac{\partial p}{\partial y}}{12 \mu} \right) - \frac{Q_{ch} \rho b^2 b^2 P_H}{8 \mu T_{eff} \beta} = \frac{\partial (\rho b h)}{\partial t},
$$

(1)

where $T_{eff}(= T_p + (3 \pi b^2/8))$ is the effective hole length [17] which includes the hole length $T_p$ and an equivalent length to account for the the end effect of the hole, $\beta = b/a$ the perfo-
The pressure difference across the hole, \( p_a \) and \( p_b \) being the pressure at the hole ends in regions I and III, respectively (see Fig. 1(d)), \( \eta(\beta) = (1 + (3b^4K(\beta)Q_{th}/16T_{eff}b^3Q_{ch})) \), and \( K(\beta) = 4\beta^2 - \beta^4 - 4 \ln \beta - 3 \). \( Q_{ch} \) and \( Q_{th} \) are the flow rate factors which account for rarefaction effect in the flow through the parallel plates and through the holes, respectively. The expression for \( Q_{ch} \) for the entire flow regime [4] and \( Q_{th} \) for the slip regime [13] are given by

\[
Q_{ch} = 1 + 3 \frac{0.01807\sqrt{\pi}}{D_0} + 6 \frac{1.35355}{D_0^{1.1468}}, \quad Q_{th} = 1 + 4Kn_{th}
\]

where \( D_0 = \sqrt{\pi}/2Kn_{ch} \), \( Kn_{ch} = \lambda/h_0 \), \( Kn_{th} = \lambda/b \) and \( \lambda = 0.0068/p_a \) at ambient temperature and pressure \( p_a \).

Eq. (1) can be linearized under the assumption of small amplitude vibration \( h = h_0 + \Delta h \) and small pressure variation \( p = p_a + \Delta p \) through the film thickness, where \( p_a \) and \( h_0 \) are constants representing the ambient pressure and the nominal gap thickness, respectively. Now, taking \( PH \approx \Delta p \) for simplicity, and introducing \( P = \Delta p/p_a \) and \( H = \Delta h/h_0 \), the linearized modified Reynolds equation for a polytropic process (i.e., \( \rho \propto p^{1/\gamma} \)), becomes

\[
\frac{\alpha^2 P}{\partial x^2} + \frac{\alpha^2 P}{\partial y^2} - \frac{P}{\alpha} = \alpha^2 \frac{\partial P}{\partial t} + \alpha^2 \nu \frac{\partial H}{\partial t}
\]

where \( l \) is the characteristic length, given by

\[
l = \sqrt{2h_0^3T_{eff}\eta(\beta)Q_{ch}/3\beta^2b^2Q_{th}}.
\]

\( \alpha^2 = 12\mu/\nu h_0^2p_aQ_{ch} \) is a constant, and \( \nu = 1 \) for isothermal processes and 1.4 for adiabatic processes at ambient conditions. The value of the characteristic length depends on different parameters related to perforations. The above governing equation may again be modified to include surface roughness effects [18].

**3. Solution procedure**

Eq. (3) is a linear, nonhomogeneous partial differential equation. It is difficult to solve this equation directly. However, we can transform it to the standard 2D diffusion equation and then solve it using Green’s function.

For transformation, we use the following mapping function [19]:

\[
P(x, y, t) = U(\xi, \gamma, t)e^{-\kappa^2t}, \quad \xi = x, \ \gamma = y, \ \kappa = \frac{1}{l\alpha}.
\]

The transformed Eq. (3), after some rearrangement, is given by

\[
\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \gamma^2} - \alpha^2 \frac{\partial U}{\partial t} = f(\xi, \gamma, t)
\]

where \( f(\xi, \gamma, t) = \nu\alpha^2 e^{-\kappa^2t}\partial H/\partial t \). Note that \( x \) and \( y \) remain the same as can be seen from the transformation function given in Eq. (4). Eq. (5) is the linear diffusion equation with a source term \( f(\xi, \gamma, t) \). The transformed equation can be solved by dif-
ferent methods such as Laplace transforms [2], Fourier series approach [20], Green’s function approach [21], etc. We use Green’s function approach in order to consider different source terms which may depend on any of the independent variables \(x, y, \) and \(t\).

### 3.1. Transverse rigid plate motion with zero pressure boundary condition on all the edges

In the following section, we solve Eq. (3), using Eqs. (4) and (5), on domain \(S\) shown in Fig. 2(a), with ambient boundary condition (i.e., \(P = 0\) on \(\delta S\)) and ambient initial condition, \(P(t = 0) = 0\), and find the squeeze film damping force and spring force for transverse motion (Fig. 2(b)) of a vibrating plate.

Let the plate, shown in Fig. 2(b), oscillate transversely with a harmonic displacement \(H(t) = \delta \exp(j \omega t)\) about the equilibrium position, i.e., \(H(0) = 0\). By solving Eq. (5) for \(U(x, y, t)\) using Green’s function and thereafter substituting it in Eq. (4), we get the non-dimensional pressure distribution \(P(x, y, t)\) as

\[
P(x, y, t) = \sum_{m,n=\text{odd}} \frac{16(-1)^{m+n-2/2}}{\pi^2 mn} \frac{-i \omega \delta \exp(j \omega t)}{\kappa^2 + k_{mn}^2/\alpha^2 + i \omega} \times \cos \frac{m \pi x}{L} \cos \frac{n \pi y}{W}
\]

where \(k_{mn} = (m^2 \pi^2/L^2) + (n^2 \pi^2/W^2)\). The total reaction force \(F(t)\) on the moving perforated plate is calculated by integrating the pressure distribution \(p_a P(x, y, t)\) over the domain \(S = \{(x, y) | -L/2 \leq x \leq L/2, -W/2 \leq y \leq W/2\}\) and then subtracting from it the force contribution due to the total area of the individual holes. The net force after normalizing it with \(L W p_a\) is given by

\[
\frac{F(t)}{L W p_a} = \sum_{m,n=\text{odd}} \frac{16}{\pi^2 mn} \frac{-i \omega \delta \exp(j \omega t)}{\kappa^2 + k_{mn}^2/\alpha^2 + i \omega} \left[4 - f_{\text{perf}} \right]
\]

(7)

Here, a negative sign shows the opposite relationship between the back force and the direction of motion. The expression for \(f_{\text{perf}}\) is given by:

- If \(N\) and \(M\) are odd

\[
f_{\text{perf}} = (-1)^{(m+n-2)/2} \sum_{i=-(N-1)/2}^{(N-1)/2} \times \sin \frac{m \pi (iq + Lh/2)}{L} \quad \text{and} \quad \sin \frac{m \pi (iq - Lh/2)}{L}
\]

\[
\times \sum_{j=-(M-1)/2}^{(M-1)/2} \times \sin \frac{n \pi (jq + Lh/2)}{W} \quad \text{and} \quad \sin \frac{n \pi (jq - Lh/2)}{W}
\]

\[
= 4(-1)^{(m+n-2)/2} \sin \frac{m \pi Lh}{2L} \sin \frac{n \pi Lh}{2W} \sum_{i=-(N-1)/2}^{(N-1)/2} \times \cos \frac{m \pi i q}{L} \sum_{j=-(M-1)/2}^{(M-1)/2} \cos \frac{n \pi j q}{W}
\]
where \( i \in \{[-(N-1)/2, (N-1)/2]: i \rightarrow i + 1 \} \) and \( j \in \{[-(M-1)/2, (M-1)/2]: j \rightarrow j + 1 \} \)

- If \( N \) and \( M \) are even

\[
\begin{align*}
f_{\text{perf}} &= (-1)^{(m+n-2)/2} \sum_{i=-(N-1)}^{N-1} \frac{m \pi (iq + Lh)}{2L} - \sin \frac{m \pi (iq - Lh)}{2L} \\
&\quad \times \sum_{j=-(M-1)}^{M-1} \frac{n \pi (jq + Lh)}{2W} - \sin \frac{n \pi (jq - Lh)}{2W} \\
&= 4 (-1)^{(m+n-2)/2} \sin \frac{m \pi Lh}{2L} \sin \frac{n \pi Lh}{2W} \sum_{i=-(N-1)}^{N-1} \frac{m \pi i q}{2L} \cos \frac{n \pi j q}{2W} \\
&\quad \times \cos \frac{m \pi i q}{2L} \sum_{j=-(M-1)}^{M-1} \cos \frac{n \pi j q}{2W}
\end{align*}
\]

where \( i \in \{[-(N-1), (N-1)]: i \rightarrow i + 2 \} \) and \( j \in \{[-(M-1), (M-1)]: j \rightarrow j + 2 \} \). Note that \( i \) and \( j \) are odd numbers.

Now, the non-dimensional spring force \( f_s \) and the damping force \( f_d \) are calculated by separating \( f_{\text{tot}} \) into real and imaginary parts, respectively. Taking the absolute value of the non-dimensional spring and damping force, we get

\[
f_s = \frac{16 \nu \delta \sigma}{\pi^8} \sum_{m,n=\text{odd}} \left[ \begin{array}{c} 4 - f_{\text{perf}} \\ ((mn)^2[ (\Gamma^2/\pi^2 + m^2 \chi^2 + n^2)^2 + \sigma^2/\pi^4] \end{array} \right]
\]

(8)

\[
f_d = \frac{16 \nu \delta \sigma}{\pi^6} \sum_{m,n=\text{odd}} \left[ \begin{array}{c} (\Gamma^2/\pi^2 + m^2 \chi^2 + n^2)^2[4 - f_{\text{perf}}] \\ ((mn)^2[ (\Gamma^2/\pi^2 + m^2 \chi^2 + n^2)^2 + \sigma^2/\pi^4] \end{array} \right]
\]

(9)

where \( \Gamma = \kappa \omega W = W/l \) is a constant which captures the perforation effect, \( \sigma = \alpha^2 W^2 \omega = 12 \mu W^2 \omega/\nu h_0^2 p_s Q_{ch} \) the well known squeeze number [2] which captures the compressibility effect, and \( \chi = W/L \) is the plate aspect ratio. Generally, \( W \) is the smallest dimension chosen out of length \( L \) and width \( W \) of the plate (in this case, we have taken width \( W \) as the smallest dimension). At the cut-off squeeze number, \( \sigma_{\text{cut-off}} \), damping and spring forces become equal. If only the leading first term is taken in the summation of Eqs. (8) and (9), then the cut-off squeeze number is approximated by

\[
\sigma_{\text{cut-off}} = \Gamma^2 + \pi^2 (\chi^2 + 1).
\]

(10)

This expression captures the essential effect of perforations on the cut-off squeeze number. The effect, as is evident from \( \Gamma^2 \) term, is rather large. The cut-off frequency increases substantially as the perforation ratio \( b/a \) increases. This is due to sudden decrease in the spring force because of perforations. Note that the second term on the right hand side is exactly the same as derived by Blech [2] for solid rectangular plates.

For a non-perforated plate, the radius of the hole, \( b \), becomes zero which implies that the characteristic length, \( l \), goes to infinity and hence \( \Gamma \) becomes zero. So, by putting \( \Gamma = 0 \) in Eqs. (8) and (9), the formulas reduce to the readily available formulas for non-dimensional spring and damping force due to squeeze film flow in a non-perforated plate [2]. Thus, Eqs. (8) and (9) can be used to calculate non-dimensional spring and damping force due to squeeze film flow under the transverse oscillations of a smooth, perforated plate with respect to another smooth fixed plate.

Assuming viscous damping and linear spring force assumption, the corresponding damping constant and spring constant can be calculated as

\[
c_a = \frac{f_a p_s \chi L^2}{\delta h_0}, \quad k_a = \frac{f_s p_s \chi L^2}{\delta h_0}.
\]

(11)

3.2. The perforation parameter \( \Gamma \)

The perforation parameter \( \Gamma = W/l \) is the ratio of the width of the plate to the characteristics length defined earlier in Section 3.1. It is the most important parameter in our analysis. It couples the effect of perforations with other effects, such as compressibility and rarefaction, in the squeeze film damping formulas derived here. Fortunately, it shows up as a distinct additional term (when compared to Blech’s formulas) in the formulas for squeeze film damping and spring force given by Eqs. (8) and (9), respectively. \( \Gamma \rightarrow 0 \) implies a non-perforated plate, whereas \( \Gamma \rightarrow \infty \) implies a heavily perforated plate, something like a sieve. So, the realistic case of a perforated plate is the finite, non-zero value of \( \Gamma \). Although, the value of \( \Gamma \) depends on many factors as mentioned earlier, it is possible to untangle it and show how different design parameters affect its value, as we show now.

Substituting the expression for the effective length \( l \), expanding \( I_{\text{eff}} \), and rearranging terms, we get

\[
\Gamma = \left( \frac{W}{h} \right) \beta \left[ \frac{2}{3} \left( \frac{T_p}{h_0} \right) \left( \frac{h_0}{b} \right)^3 R_Q + \left( \frac{\pi}{4} \right) \left( \frac{h_0}{b} \right)^3 R_Q + \frac{1}{8} \kappa(\beta) \right]^{1/2}
\]

(12)

where \( R_Q = Q_{ch}/Q_{th} \) is the ratio of the rarefaction flow factor in the air-gap between the plates to that through the holes. This expression for \( \Gamma \) (Eq. (12)) shows that the value of \( \Gamma \) depends on three terms. The first term represents the flow resistance through the holes which we call the hole-effect. The second term is due to the edge effect of the holes, so we call it the edge-effect. The last term is solely due to the region under a cell, so we call it the cell-effect. Moreover, the first two terms also depend on the rarefaction ratio \( R_Q \). Fig. 3 shows the relative effect of different terms on \( \Gamma \) for \( \beta \in (0, 1) \) at \( W = 200 \mu m, T_p = 2 \mu m, h_0 = 2 \mu m, \) and \( b = 5 \mu m \). The figure shows that the term representing the cell-effect dominates over the edge and the hole-effect for \( \beta < 0.4 \). On the other hand, the cell-effect, captured by \( K(\beta) \), can be neglected for \( \beta > 0.8 \). Thus, we get three distinct ranges of \( \beta \) that have qualitatively distinct effects on \( \Gamma \):
• **Cells dominate.** For $\beta < 0.4$ and large hole radius compared to the gap thickness, we have $K(\beta)/8 \gg (2/3)(h_0/b)^3[(T_p/b) + (3\pi/8)] R_Q$ for reasonable values of the plate thickness. In this case, $\Gamma \approx (W/b)\beta \sqrt{8/K(\beta)}$. Thus, the value of perforation parameter in this case is governed by the planar geometry of holes and their pitch.

• **Holes dominate.** For $\beta > 0.8$, the holes are so close together that the cell-effect gets completely overwhelmed by the action near the holes and in the holes. In this case, $K(\beta)$ is negligible, and therefore, the perforation parameter is given by $\Gamma \approx (W/b)\beta \sqrt{8/K(\beta)}$, which scales linearly with $\beta$ for a given hole radius.

• **Mixed effects.** For $0.4 < \beta < 0.8$, both cells and holes have comparable effect on $\Gamma$. This is the intermediate range where we cannot ignore the effect of either. For comparable hole radius, gap and plate thickness, Fig. 4 shows the variation of $\Gamma$ with $\beta$ at different hole radii for the above mentioned values of $W$, $T_p$, and $h_0$. It shows that the same value of $\Gamma$ can be obtained for two different topologies of the perforation geometry. For example, $\Gamma = 66.8$ is the same for $b = 9 \text{ \mu m}$ and $b = 3 \text{ \mu m}$ at $\beta = 0.63$. This provides flexibility in design where the same macroscopic dynamic characteristics from the squeeze film can be obtained with different hole sizes and pitch.

### 3.3. Effect of flexibility on the rectangular plate fixed at two parallel edges

In the preceding discussions, we have assumed the moving plate to be rigid. However, with anchored or partially anchored edges, 2D MEMS structures flex during transverse motion. We now consider the effect of non-rigidity of such structures on the squeeze film damping calculations discussed above. As is evident from the solution, the most critical quantity in deriving the result is the determination of pressure distribution under the plate. When we consider a flexible plate in transverse motion, the boundary conditions and the mode shape affect this pressure distribution. However, both these quantities are problem specific. Therefore, one cannot derive an analytical solution for flexible plates that can take care of all boundary conditions and mode shapes. Here, we consider a specific example and derive the result for an assumed mode shape to show the steps in such derivations. For a rectangular plate fixed at the two parallel edges, let us assume the deflected shape participating in the transverse motion to be $\Psi(x, y) = (16/L^4)((L/2) - x)^2((L/2) + x)^2$ as shown in Fig. 2(d). If the displacement amplitude at the middle is taken as the generalized co-ordinate $Z(t)$, then the displacement may be expressed as [22]

$$H(x, y, t) = \Psi(x, y)Z(t) = \frac{16}{L^4} \left( \frac{L}{2} - x \right)^2 \left( \frac{L}{2} + x \right)^2 Z(t)$$

(13)

about the equilibrium position. Assuming sinusoidal motion $Z(t) = \delta e^{i\omega t}$ and solving Eq. (3) with zero pressure condition on the free edges and zero flow condition on the fixed edges, we get the following expression for non-dimensional pressure distribution $P^s(x, y, t)$:

$$P^s(x, y, t) = \sum_{n=odd}^{2\pi} \frac{32(-1)^{(n-1)/2}}{15\pi n} \frac{\delta e^{i\omega t} \cos n\pi y}{\omega^2 + k_n^2/\alpha^2 + i\omega}$$

(14)

where $k_n = n^2 \pi^2 / W^2$. Integrating over the perforated area and separating the result into its real and imaginary part, we find out the following expression of the spring force $f_s$ and the damping force $f_d$:

$$f_s = \frac{512\pi^2\delta^2}{225\pi^6} \sum_{n=odd} \frac{[1 - f_{\text{perf}}]}{n^2([\Gamma^2/\pi^2 + n^2]^2 + 4\alpha^2/\pi^4)}$$

(15)
\[ f_\mu^* = \frac{512 \nu \sigma}{225 \pi^4} \sum_{n=odd} (\Gamma^2/\pi^2 + n^2)[1 - f_{\text{perf}}] \]  
\[ = \frac{n^2((\Gamma^2/\pi^2 + n^2)^2 + \sigma^2/\pi^4)}{n^2(\Gamma^2/\pi^2 + n^2)^2 + \sigma^2/\pi^4} \]

The expression for \( f_{\text{perf}} \) are as follow:

- If \( N \) and \( M \) are odd
  \[ f_{\text{perf}} = \frac{15(-1)^{(n-1)/2}}{8} \sum_{i=-(N-1)/2}^{(N-1)/2} \left[ R_y - R_x \right] \frac{8 R_y^3 - 3 R_x^3 - 16 R_y^5 - R_x^5}{3 L^3} \frac{n \pi j q}{W} \]

- If \( N \) and \( M \) are even
  \[ f_{\text{perf}} = \frac{15(-1)^{(n-1)/2}}{8} \sum_{i=-(N-1)/2}^{(N-1)/2} \left[ R_y - R_x \right] \frac{8 R_y^3 - 3 R_x^3 - 16 R_y^5 - R_x^5}{3 L^3} \frac{n \pi j q}{W} \]

Finally, the generalized damping and spring constants with respect to the generalized coordinate \( Z(t) = \delta \text{Real}(e^{i\omega t}) \) at the middle of the plate are given by

\[ f_\mu^* = \frac{f_\mu^*}{\delta \alpha h_0} \quad k_\mu^* = \frac{k_\mu^*}{\delta \alpha h_0} \quad (17) \]

For small amplitude motion, the mode shape of the plate has much smaller effect on the pressure distribution compared to the effect of boundary conditions. Therefore, it is worthwhile to consider the rigid plate solution which is simple and elegant with modified boundary conditions for the fluid domain and compare the results with those obtained for the flexible plate.

### 3.4. Transverse rigid plate motion with zero pressure boundary condition on free edges and zero-flow condition on fixed edges

In this section, in order to capture the effect of flexibility on the squeeze film damping, we compare the damping and the spring constant in the flexible and the rigid plates, respectively, for the same boundary conditions. Here, we point out that the two fixed edges considered for the rigid plate case refer to the solution of the fluid flow equation in the fluid domain with commensurate boundary conditions on pressure and its gradient. While the flexible plate solution with the two parallel fixed edges is obtained in the preceding section, the solution for the rigid plate with the same boundary conditions is discussed in this section.

For solving this case which is also shown in Fig. 2(b) and (c), we take zero flow condition on the fixed edges and zero pressure condition on the free edges. Assuming rigid plate motion with the above boundary conditions, we solve Eq. (3) in the same way as discussed in Section 3.1, we get the following expression for the non-dimensional pressure, the spring force \( f_s \) and the damping force \( f_d \):

\[ P(x, y, t) = \sum_{n=odd} \frac{4(-1)^{(n-1)/2}}{\pi n} \frac{-i\omega \delta e^{i\omega t}}{k_n^2 + k_n^2/\alpha^2 + i\omega} \frac{n \pi j q}{W} \]

\[ f_s = \frac{8 \nu \sigma^2}{\pi^6} \sum_{n=odd} \frac{[1 - f_{\text{perf}}]}{n^2((\Gamma^2/\pi^2 + n^2)^2 + \sigma^2/\pi^4)} \]

\[ f_d = \frac{8 \nu \sigma^2}{\pi^4} \sum_{n=odd} \frac{[(\Gamma^2/\pi^2 + n^2)^2 + \sigma^2/\pi^4]}{n^2((\Gamma^2/\pi^2 + n^2)^2 + \sigma^2/\pi^4)} \]

The expression for \( f_{\text{perf}} \) are as follow:

- If \( N \) and \( M \) are odd

\[ f_{\text{perf}} = \frac{(-1)^{(n-1)/2} L_h}{2} \frac{(N-1)/2}{(M-1)/2} \sum_{i=-(N-1)/2}^{(N-1)/2} \sum_{j=-(M-1)/2}^{(M-1)/2} \frac{n \pi j q}{W} \frac{n \pi L_h}{2W} \]

- If \( N \) and \( M \) are even

\[ f_{\text{perf}} = \frac{(-1)^{(n-1)/2} L_h}{2} \frac{(N-1)/2}{(M-1)/2} \sum_{i=-(N-1)/2}^{(N-1)/2} \sum_{j=-(M-1)/2}^{(M-1)/2} \frac{n \pi j q}{W} \frac{n \pi L_h}{2W} \]

where \( i \in \{-(N-1), (N-1): i \rightarrow i+1 \} \) and \( j \in \{-(M-1), (M-1): j \rightarrow j+1 \} \). Note that \( i \) and \( j \) are odd numbers.

The spring and damping constants can also be obtained using Eq. (11).

Now, we compare the spring and damping constants for rigid and flexible rectangular plates fixed at the two parallel edges for the same maximum vibrational amplitude and boundary conditions. For the rigid plate, we use Eqs. (19), (20) and (11), while for the flexible plate, we use Eqs. (15)–(17). Assuming zero perforation so that both \( \Gamma \) and \( f_{\text{perf}} \) become zero, we get the
following relationship:

\[ R_c \equiv \frac{c_{\text{flex}}}{c_{\text{rigid}}} = \frac{64}{225}, \quad R_k \equiv \frac{k_{\text{flex}}}{k_{\text{rigid}}} = \frac{64}{225} = R_c \] (21)

Hence, one can obtain the damping and spring constant for the flexible plate by multiplying the rigid plate formulae with some multiplication factor \( R_c \) and \( R_k \). Note that the formulae in both the cases are obtained for the same boundary conditions. For the case described above, we get \( R_c = R_k = 64/225 \). So, we state that the flexibility in this case reduces the damping and the spring constants by a factor of about 3.5. Later, we also show that this multiplication factor holds equally good for the perforated plate without any substantial error.

### 3.5. Maximum frequency range

Since the main assumption in the formulation is the negligence of inertial effect, we need to figure out the limits on dynamic operations of these structures which keep this assumption reasonably intact. One such measure is that the Reynolds number should be very low. From the order of magnitude analysis of the equation of squeeze-film flow through parallel plates and the equation of flow through the holes, we get the following condition

- For inertialess flow in the gap between the plates, the modified Reynolds number \( \text{Re} = \rho \omega h^2 Q_{ch}/\mu \) is less than one. The maximum frequency which satisfies this condition is given by \( f_1 \ll \mu/(2\pi \rho h^2 Q_{ch}) \).
- For inertialess flow through the holes, the modified Reynolds number \( \text{Re} = \rho \omega b^2 Q_{ch}/\mu \) is less than one. Therefore, the maximum frequency below which the above conditions satisfy is given by \( f_2 \ll \mu/(2\pi \rho b^2 Q_{in}) \).

Therefore the maximum allowable frequency \( f_{\text{max}} \) below which the inertia effect is neglected both in the air gap between the plates and the holes should be the minimum of \( f_1 \) and \( f_2 \). Hence, the expression of \( f_{\text{max}} \) is given by \( \text{min}(f_1, f_2) \).

### 4. Validation and discussion

In the preceding section, we showed that the derived formulae for transverse motion of a rigid perforated plate reduces to the conventional formulae [2] for non-perforated case when \( \Gamma \to 0 \). In this section, we validate the analytical results with the solution of 3D Navier–Stokes equation solved using ANSYS at different perforation ratios. For simplicity, we consider the isothermal air-flow, i.e., \( \nu = 1 \), although adiabatic flow conditions pose no serious computational difficulties.

For the purpose of validation, we consider the following dimensions of the plate and the holes: \( L = 100 \mu m, W = 50 \mu m, h_0 = 4 \mu m, T_p = 2 \mu m, N = 8, M = 4, q = 12.5 \mu m \). We keep \( L_h \), the size of each square hole, as a variable parameter. The properties of air, which is used as the fluid medium, are \( \mu = 1.8 \times 10^{-5} \text{Ns/m}^2, \rho = 1.2 \text{kg/m}^3 \) at temperature \( T = 293 \text{K} \) and pressure \( p_0 = 1 \text{atm} \) (= 1.013 \times 10^5 \text{Pa})). Here, we take \( L_h = 0.5, 7.5, 8.75 \) and 10 \( \mu m \) which gives the perforation ratio \( \beta = L_h/q = 0.0, 0.4, 0.6, 0.7 \) and 0.8. The corresponding maximum frequencies as explained in Section 3.5 are approximately 138, 138, 132, 98 and 75 kHz, respectively. So, the operating frequency must be less than the corresponding maximum limit to avoid significant error. In the validation of different formulae, we use a frequency of 500 Hz where spring and inertial effects are negligible.

#### Table 1

<table>
<thead>
<tr>
<th>Perforation ratio, ( \beta )</th>
<th>Analytical result</th>
<th>Numerical result</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.2356E-006</td>
<td>2.3623E-006</td>
<td>5.36</td>
</tr>
<tr>
<td>0.4</td>
<td>9.6624E-007</td>
<td>9.6731E-007</td>
<td>0.11</td>
</tr>
<tr>
<td>0.6</td>
<td>3.2370E-007</td>
<td>3.1351E-007</td>
<td>3.25</td>
</tr>
<tr>
<td>0.7</td>
<td>1.7184E-007</td>
<td>1.6785E-007</td>
<td>2.38</td>
</tr>
<tr>
<td>0.8</td>
<td>8.2443E-008</td>
<td>8.1289E-008</td>
<td>1.42</td>
</tr>
</tbody>
</table>

The perforation ratio is defined as \( \beta = L_h/q \). The percentage error is computed by normalizing the absolute error with the numerical value.

#### 4.1. Rigid plate performing transverse motion

In order to do comparative study and validation, we carry out numerical simulations in ANSYS for the pressure and force calculations for different values of \( L_h \) the size of the hole. In ANSYS, we first build a solid model of the fluid domain, mesh it with FLUID 142 elements, apply zero pressure boundary condition on the free boundaries of the plate and open end of the perforation above the perforation; and apply zero velocity condition on the perforated plate, while moving velocity condition is applied on the lower plate. The total back force on the perforated plate is calculated by integrating the back pressure over the perforated area. Thereafter, the damping constant can be calculated by dividing the total back force as shown in Eq. (11). In Table 1, we compare the numerically calculated damping constant with the analytical result obtained from Eq. (11) for a structure vibrating at 500 Hz. Fig. 5 shows the pressure distribution and velocity vectors for a perforation factor of 0.7. It shows a quite nice agreement over the perforation ratio of 0.8 as presented in Table 1.

The mean error in this case is about 0.3%. The error enlisted in Table 1 is due to various numerical factors such as the mesh density, consideration of air with the same viscosity in the gap between the plates and the perforations whose characteristics flow length is the hole radius, negligence of miscellaneous effects in intermediate region [13], etc.

#### 4.2. Effect of flexibility

In this case, we use zero flow condition on the fixed edges and zero pressure condition on the free edges unlike the previous case in which we used zero pressure boundary condition on all the four edges. As we have seen in Section 3.4, the effect of flexibility on the non-perforated rectangular plate fixed at two parallel edges can be incorporated into the rigid plate formula...
of squeeze film damping by just multiplying it with the multiplication factor $R_c = 64/225$.

Table 2 compares the analytical results for the flexible plate with those for the rigid plate with and without multiplication factor which accounts for flexibility effect on the squeeze-film damping. The percentage error at different perforation ratios is less than 1%. It shows that one can use the rigid plate solution but with some correction term to calculate equivalent squeeze-film damping in flexible structures for the same degree of perforations and boundary conditions.

5. Conclusions

In this paper, we show that the incompressible modified Reynolds equation, which is modified to include perforation effects, can be extended to include low and medium compressibility effect and the effect of perforations for the calculation of fluid damping due to flow through holes and squeeze flow of air in a thin gap between two parallel plates. The maximum frequency range up to which the formulae capture the compressibility effect is limited by the negligence of inertia effect which is discussed in Section 3.5. Before solving the modified Reynolds equation, we transform it to the standard form of the 2D diffusion equation using a simple mapping function. Then, we solve the transformed form of the equation using Green’s function and find closed form expressions for the squeeze-film damping and spring force. These expressions contain a term $\Gamma$ that captures the effect of perforations along with rarefaction effects. The formulae for rigid plate hold for the two extreme conditions as well—the incompressible case and the non-perforated case which are validated with numerical results obtained by solving 3D Navier–Stokes equation using ANSYS. Subsequently, we derive the closed form solution for a flexible plate vibrating in its first mode. It is found that for the same degree of perforation and boundary conditions, the formula for flexible plate can be approximated by just multiplying the rigid plate solution with a multiplication factor. For rectangular plate fixed at the two parallel edges, the multiplication factor comes out to be $64/225$ with negligible error ($\ll 1\%$).

We also state that the perforation factor defined in this paper is very useful to characterize the effect of various parameters of perforations. Finally, we conclude that the formulae obtained for rigid and flexible plate can be used without any substantial error.

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References


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